Finite element method in 1D

Non homogeneous Dirichlet boundary conditions

Neumann boundary conditions

Advection diffusion equation

Lucia Gastaldi

DICATAM - Sez. di Matematica,
http://lucia-gastaldi.unibs.it

UNIVERSITÀ
DEGLI STUDI
DI BRESCIA

Outline

Non homogeneous Dirichlet boundary conditions

Problem Find u such that

$$-\mu u'' + \sigma u = f \quad \text{in }]a, b[\qquad u(a) = \alpha, \; u(b) = \beta$$

Let us consider $Rg \in H^1$ such that $Rg(a) = \alpha$ and $Rg(b) = \beta$ (e.g. $Rg(x) = \alpha \frac{b-x}{b-a} + \beta \frac{x-a}{b-a}$) and $u_0 = u - Rg$. Hence $u_0 \in H^1_0(a, b)$ solves:

$$-\mu u_0'' + cu_0 = -\mu u'' + cu + \mu Rg'' - cRg = f + \mu Rg'' + cRg$$

$$u_0(a) = u_0(b) = 0.$$

Weak formulation

Find $u_0 \in H^1_0(a, b)$ such that

$$a(u, v) = F(v) - a(Rg, v) \quad \forall v \in H^1_0(a, b),$$

where $\quad a(u, v) = \int_a^b (\mu u'v' + \sigma uv)\, dx, \quad F(v) = \int_a^b fv\, dx$

Non homogeneous Dirichlet boundary conditions (in practice)

We consider Rg_h such that $Rg_h(a) = \alpha$ and $Rg_h(b) = \beta$.
E.g.: $Rg_h(x) = \alpha\varphi_a(x) + \beta\varphi_b(x)$ with φ_a and φ_b basis functions associated to the end points a and b, respectively.

$$a(Rg_h, \varphi_i) = \alpha a(\varphi_a, \varphi_i) + \beta a(\varphi_b, \varphi_i) = B\vec{Rg}.$$

Here $B \in \mathbb{R}^{ndof \times 2}$ with elements

$$B_{ij} = \mu \int_a^b \varphi_j' \varphi_i' \, dx + \sigma \int_a^b \varphi_j \varphi_i \, dx \quad j = a, b \text{ e } i = 1, \ldots, ndof.$$

We compute $u_{0h} \in V_h$ as follows:

$$A\vec{u_0} = b - B\vec{Rg}, \qquad \vec{u} = \vec{u_0} + \vec{Rg}.$$

Notice that the vector $-B\vec{Rg}$ has only the first and the last components different from zero with values

$$-B\vec{Rg}_1 = \alpha(\mu/h + h\sigma), \quad -B\vec{Rg}_{ndof} = \beta(\mu/h + h\sigma).$$

Exercize 5

Consider the following differential equation

$$-u'' + 2u = 2e^{-x}(\cos(x) - \sin(x) \quad x \in (0, 2\pi)$$
$$u(0) = 1, \quad u(2\pi) = e^{-2\pi}.$$

with exact solution $u(x) = e^{-x}\cos(x)$.

- The function `femP1` is a simplified version of `fem1d` and computes the solution only with linear piecewise polynomials. Modify the function `femP1` in order to take into account non homogeneous boundary conditions.
- Solve the differential equation by means of the modified function `femP1` with `N=[10 20 40 80 160 320]`.
- Compute the relative error in the L^2-norm of the function and of its derivative using the function `errore`.
- Plot the error in a bilogarithmic scale.

Neumann boundary conditions

Let us consider the following differential equation

$$-\mu u'' + \sigma u = f \quad \text{in } [a,b]$$
$$u'(a) = \alpha, \qquad u'(b) = \beta$$

We multiply the equation by $v \in H^1(a,b)$ and integrate on (a,b)

$$\int_a^b (-\mu u'' + \sigma u) v dx = [-\mu u' v]_a^b + \int_a^b (\mu u' v' + \sigma u v) dx$$

$$= -\mu u'(b) v(b) + \mu u'(a) v(a) + \int_a^b (\mu u' v' + \sigma u v) dx$$

$$= -\mu \beta v(b) + \mu \alpha v(a) + \int_a^b (\mu u' v' + \sigma u v) dx = \int_a^b f v dx$$

Weak formulation

Find $u \in H^1(a,b)$ such that

$$a(u,v) = F(v) + \mu \beta v(b) - \mu \alpha v(a) \quad \forall v \in H^1(a,b).$$

Piecewise linear finite elements discretization

We subdivide $[a, b]$ into N intervals with size $h = (b - a)/N$

$$V_h = \{v \in H^1(a, b) : v|_{I_i} \in \mathbb{P}^1\}$$

hence we have $N + 1$ basis functions φ_i including those associated to the end points. The matrix are:

$$K = \frac{1}{h}\begin{bmatrix} 1 & -1 & \dots & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & \dots & -1 & 1 \end{bmatrix}$$

$$M = \frac{h}{6}\begin{bmatrix} 2 & 1 & \dots & \dots & \dots & \dots & 0 \\ 1 & 4 & 1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 4 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 1 & 4 & 1 \\ 0 & \dots & \dots & \dots & \dots & 1 & 2 \end{bmatrix}$$

- If $\sigma > 0$ we find the solution by solving the linear system $(\mu K + \sigma M)u = \underline{\mathbf{F}}$.
- If $\sigma = 0$ the following compatinility condition is necessary for the solvability of the system

$$\int_a^b f dx + \mu\beta - \mu\alpha = 0$$

and we compute a solution with zero mean value.
In practice we assign the value of u in an internal node of the mesh and then we correct the mean value.

Exercize 6

$\sigma > 0$

Let us consider the following differential equations:

$$
\begin{array}{l}
-u''(x) + 2u(x) = e^x \quad \text{for } x \in [0,1] \\
u'(0) = 1, \quad u'(1) = e \\
u(x) = e^x.
\end{array}
$$

$$
\begin{array}{l}
-u''(x) + u(x) = -2 + 12x - 12x^2 + x^2(x-1)^2 \text{ for } x \in [0,1] \\
u'(0) = 0, \quad u'(1) = 0 \\
u(x) = x^2(x-1)^2.
\end{array}
$$

Solve the above differential equations using the function `femP1N` and values

`N=[8 16 32 64 128 256 512]`

Compute the error by using function `errore`.

Exercize 7

$\sigma = 0$

Let us consider the following differential equations:

$$
\begin{array}{l}
-u''(x) = -e^x \quad \text{per } x \in [0,1] \\
u'(0) = 1, \quad u'(1) = e \\
u(x) = e^x - e + 1.
\end{array}
$$

$$
\begin{array}{l}
-u''(x) = -2 + 12x - 12x^2 \text{ per } x \in [0,1] \\
u'(0) = 0, \quad u'(1) = 0 \\
u(x) = x^2(x-1)^2 - 1/30.
\end{array}
$$

Solve the above differential equations using the function `femP1N` and values

`N=[8 16 32 64 128 256 512]`

.

Compute the error by using function `errore`.

Advection diffusion reaction equation

$$
\begin{aligned}
-\mu u'' + \beta u' + cu = f \quad & \text{in } (a, b) \\
u(a) = u_a, \quad u(b) = u_b &
\end{aligned}
$$

Weak problem

Find $u \in H_0^1(a, b)$ such that

$$\int_a^b (\mu u' v' + \beta u' v + cuv)\, dx = \int_a^b fv\, dx \quad \forall v \in H_0^1(a, b)$$

Construction of the matrices

It remains to compute the matrix relative to the first order term. We subdivide the interval $[a, b]$ in N equal parts of length $h = (b - a)/N$, the nonsymmetric matrix associated with the first order term

$$B = \frac{1}{2} \begin{bmatrix} 0 & 1 & \dots & \dots & \dots & \dots & 0 \\ -1 & 0 & 1 & \dots & \dots & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & -1 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & \dots & -1 & 0 & 1 \\ 0 & \dots & \dots & \dots & \dots & -1 & 0 \end{bmatrix}$$

Exercize 8

Let us consider the differential equation

$$-u'' + 2u' = f \quad x \in (0,1) \qquad u(0) = u_a,\ u(1) = u_b$$

with the following data:

$$f_1(x) = 4 - 4x,\ u_a = u_b = 0$$
$$f_2(x) = -2(1-x)^2 - 2x^2 + 4 + 4x(1-x)(3-2x),\ u_a = 0,\ u_b = 2$$
$$f_3(x) = (4\pi^2 - 4\pi)\sin(2\pi x) + (4\pi^2 + 4\pi)\cos(2\pi x)\ u_a = u_b = 1$$
$$f_4(x) = e^x(1-x)\ u_a = 1,\ u_b = 0$$

Risolvere l'equazione differenziale con elementi finiti lineari e quadratici usando la function `fem1d` al variare di `N=[10 20 40 80 160 320]`.
Calcolare l'errore relativo in norma L^2 sia della funzione che della derivata usando la function `errore`.
Riportare opportunamente i risultati in scala bilogaritmica.

Exact solutions of exercize 8

$$
\begin{aligned}
u_1(x) &= x(1-x) \\
u_2(x) &= x^2(1-x)^2 + 2x \\
u_3(x) &= \sin(2\pi x) + \cos(2\pi x) \\
u_4(x) &= e^x(1-x)
\end{aligned}
$$

Advection dominated problems

We consider the differential equation

$$\begin{array}{l} -\varepsilon u''(x) + u'(x) = 0 \quad \text{per } x \in [0,1] \\ u(0) = 0, \quad u(1) = 1. \end{array}$$

Solve the equation by means of linear finite elements with `N=[10 20 40 80 160]`, and $\varepsilon = 1$, $\varepsilon = 0.1$ e $\varepsilon = 0.01$.