Finite element method in one dimension

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Outline

1. Second order differential equation
   - Weak formulation
   - Galerkin's method
2. Finite elements
   - Assembling the matrix and the right hand side
   - Boundary conditions
   - Exercises

Matlab Files at the link: https://lucia.gastaldi.unibs.it/fem1D

Dirichlet problem

Problem in one dimension

Let

- $\Omega =]a, b[$,
- $\mu \in \mathbb{R}$, with $\mu > 0$
- $\sigma : \Omega \to \mathbb{R}$ such that $0 \leq \sigma \leq \overline{\sigma}$
- $f : \Omega \to \mathbb{R}$

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$-(\mu u'(x))' + \sigma u(x) = f(x) \quad \text{per } x \in \Omega$$
$$u(a) = \alpha, \quad u(b) = \beta.$$

Problem with homogeneous Dirichlet boundary conditions

Problem

$$\begin{aligned} &-\mu u'' + \sigma u = f \quad \text{in } \Omega =]a,b[\\ &u(a) = u(b) = 0 \end{aligned}$$

Functional setting

We introduce the following Hilbert spaces

$$\mathrm{L}^2(a,b) = \{v : (a,b) \to \mathbb{R} : \int_a^b v^2\,dx < +\infty\}$$

$$H^1(a,b) = \{v \in \mathrm{L}^2(a,b) : v' \in \mathrm{L}^2(a,b)\}$$

$$V = H_0^1(a,b) = \{v \in H^1(a,b) : v(a) = v(b) = 0\}$$

endowed with the following norms

$$\|v\|_0 = \left(\int_a^b |v|^2\,dx\right)^{1/2}, \quad \|v\|_1 = \left(\|v\|_0^2 + \|v'\|_0^2\right)^{1/2}$$

We multiply the equation by $v \in V$ (test function) and integrate on (a, b):

$$\int_a^b (-\mu u''(x) + \sigma u(x)) v(x)\, dx = \int_a^b f(x) v(x)\, dx$$

By integration by parts we have

$$\int_a^b \mu u''(x) v(x)\, dx = \big[\mu u'(x) v(x)\big]_a^b - \int_a^b \mu u'(x) v'(x)\, dx$$

from which

$$\int_a^b (\mu u'(x) v'(x) + \sigma u(x) v(x))\, dx = \int_a^b f(x) v(x)\, dx$$

We set

$$a(u, v) = \int_a^b (\mu u'(x) v'(x) + \sigma u(x) v(x))\, dx, \quad F(v) = \int_a^b f(x) v(x)\, dx$$

Weak Problem

Find $u \in V$ such that

$$a(u, v) = F(v) \quad \forall v \in V.$$

Lax-Milgram Lemma

Assume that the bilinear form a is continuous and coercive and that the linear functional F is bounded, that is

$$\exists M_a > 0 \text{ s.t. } a(u, v) \leq M_a \|u\|_V \|v\|_V$$
$$\exists \alpha > 0 \text{ s.t. } a(u, u) \geq \alpha \|u\|_V^2$$
$$F(v) \leq \|F\|_{V'} \|v\|_V$$

Then there exists a unique solution $u \in V$ of the weak problem. Moreover, the following a priori estimate holds true

$$\|u\|_V \leq \frac{\|F\|_{V'}}{\alpha}.$$

Galerkin's method

We consider a finite dimensional subspace V_h of V.

$$V_h \subset V, \qquad \dim V_h = N(h) < +\infty$$

Discrete problem

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Let $\{\varphi_1, \ldots, \varphi_{N(h)}\}$ be a basis for V_h, hence $u_h = \sum_{j=1}^{N(h)} u_j \varphi_j$.

The vector $\underline{u} \in \mathbb{R}^{N(h)}$ with components u_j satisfies

$$a\left(\sum_{j=1}^{N(h)} u_j \varphi_j, \varphi_i\right) = F(\varphi_i) \text{ for all } i = 1, \ldots, N(h).$$

Galerkin's method

Since a is bilinear we have

$$\sum_{j=1}^{N(h)} u_j a(\varphi_j, \varphi_i) = F(\varphi_i) \quad i = 1, \dots, N(h).$$

For simplicity, let us assume that σ is a constant function and define the following matrices with elements

$$K_{ij} = \int_a^b \varphi_j' \varphi_i' \, dx \qquad M_{ij} = \int_a^b \varphi_j \varphi_i \, dx,$$

and the vector $\underline{\mathrm{F}}$ with components

$$\underline{\mathrm{F}}_i = \int_a^b f \varphi_i \, dx.$$

Setting $A = \mu K + \sigma M$, the discrete problem is equivalent to the following **linear system**

$$A\underline{\mathrm{u}} = \underline{\mathrm{F}}$$

with A symmetric and positive definite.

Error estimates

We recall the definition of the norms

$$\|v\|_0 = \left(\int_a^b v^2\,dx\right)^{1/2}, \quad \|v\|_V = \left(\|v\|_0^2 + \|v'\|_0^2\right)^{1/2},\ \forall v \in V$$

Céa's Lemma

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v \in V_h} \|u - v_h\|_V.$$

The error is bounded by the best approximation in V_h.
We need a good choice for V_h!

Finite elements in 1D

1) domain: interval
2) space: $\mathbb{P}_r$
3) d.o.f.: depend on the degree of polynomials
Piecewise linears: endpoints (2)
Piecewise quadratics: endpoints + midpoint (3)
...

Approximation properties

Let $\Pi^1_h u(x) = \sum_{i=1}^{N(h)} u(x_i)\varphi_i(x)$ be the piecewise linear interpolant of u then

$$\inf_{v_h \in V_h} \|u - v_h\|_0 \le \|u - \Pi^1_h u\|_0 \le h^2 \|u''\|_0$$

$$\inf_{v_h \in V_h} \|u' - v_h'\|_0 \le \|u' - (\Pi^1_h u)'\|_0 \le h \|u''\|_0$$

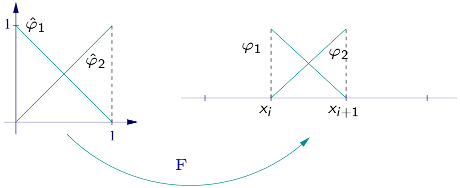
Stiffness and mass matrices

K stiffness matrix, M mass matrix with the following structure:

$$K = \frac{1}{h}\begin{bmatrix} 2 & -1 & \dots & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & \dots & -1 & 2 \end{bmatrix}$$

$$M = \frac{h}{6}\begin{bmatrix} 4 & 1 & \dots & \dots & \dots & \dots & 0 \\ 1 & 4 & 1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 4 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & 1 & 4 & 1 \\ 0 & \dots & \dots & \dots & \dots & 1 & 4 \end{bmatrix}$$

Reference element and current element



$\hat{K} = [0, 1]$
reference element
$K = I_i = [x_{i-1}, x_i]$
current element

$\hat{\varphi}_1, \hat{\varphi}_2$ basis functions on $\hat{K}$;
φ_1, φ_2 basis functions on K.

$$F_K : \hat{K} \to K, \quad x = F_K(\hat{x})$$
$$\varphi(x) = \hat{\varphi}(F_K^{-1}(x))$$

Hence:

$$x = F_K(\hat{x}) = x_{i-1} + h\hat{x}.$$

Computing the right hand side

We have

$$\underline{F}_i = F(\varphi_i) = \int_\Omega f(x)\varphi_i(x)dx = \sum_K \int_K f(x)\varphi_i(x)dx.$$

By a change of variable one gets

$$\begin{aligned}\int_K f(x)\varphi_i(x)dx &= \int_{\hat{K}} f(F_K(\hat{x}))\hat{\varphi}_i(\hat{x})F_K'(\hat{x})\,d\hat{x} \\ &= h\int_{\hat{K}} f(F_K(\hat{x}))\hat{\varphi}_i(\hat{x})\,d\hat{x}\end{aligned}$$

In order to compute this integral we need suitable quadrature formula so that the quadrature error is of *higher order* with respect to the interpolation error.
For example, one can use trapezoidal rule, Cavalieri-Simpson rule or Gauss-Legendre quadrature formula.

Gauss - Legendre formula

Let n denote the degree of polynomials used to construct the formula. The nodes $\tilde{x}_i$ and the weights $\tilde{w}_i$ are given in $[-1, 1]$.

n	nodes $\tilde{x}_i$ $i = 0, \dots, n$	weights $\tilde{w}_i$ $i = 0, \dots, n$
0	(0)	(2)
1	$(-1/\sqrt{3}, 1/\sqrt{3})$	$(1, 1)$
2	$(-\sqrt{15}/5, 0, \sqrt{15}/5)$	$(5/9, 8/9, 5/9)$

n	D.o.P.	order
0	1	$CH^2 \max \lvert f^{(2)} \rvert$
1	3	$CH^4 \max \lvert f^{(4)} \rvert$
2	5	$CH^6 \max \lvert f^{(6)} \rvert$

Nodes $\hat{x}_i$ and weights $\hat{w}_i$ on $[0, 1]$ can be obtained with the mapping:

$$\hat{x}_i = \frac{1 + \tilde{x}_i}{2}, \quad \hat{w}_i = \tilde{w}_i.$$

Assembling the right hand side

- Loop on the elements $ie = 1, \dots, ne$
- Compute the **local** right hand side $F_i^{loc} = F(\varphi_i)$, $i = 1, \dots, ndof$
- Loop on the local d.o.f $i = 1, \dots, ndof$ and assembling the **global** right hand side $F_{iglob} = F_{iglob} + F_i^{loc}$
- Enforcing boundary conditions

Homogeneous Dirichlet boundary conditions

The right hand side constructed so far has two more components corresponding to the basis functions associated to the end points.

$$\underline{F}_{prima} = \begin{pmatrix} 0.1 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \\ 0.1 \end{pmatrix}$$

We eliminate the first and the last component:

$$\underline{F}_{dopo} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}$$

Function `fem1d`

```
[x,u]=fem1d(mu,f,sigma,a,b,N,k)
```

Input

`mu`	coefficient of the second derivative
`sigma`	function in front of u
`f`	right hand side
`a,b`	end points
`N`	number of the intervals
`k`	degree 1 or 2

Output

`x`	nodes
`u`	solution

Main steps

- Basis functions on the reference element `shape.m`
- Construction of the mesh
  - compute h
  - compute the nodes
- Construction of the matrix and the right hand side
  - construct matrices `K, M` with the sparse format
  - construct the right hand side `carico.m`
    Cicle on the elements
    - compute the local right hand side
    - assemble of the right hand side
- Boundary conditions
- Solution of the linear system
- Output `x,u`

Error

If we know the exact solution, the function `errore` provides the quantities:

$$\|u - u_h\|_0 \qquad \|u' - u_h'\|_0.$$

```
[E0,E1]=errore(u,a,b,esatta,desatta,N,k)
```

Input

`u`	solution obtained using `fem1d`
`a,b`	interval endpoints
`esatta, desatta`	functions-handle with the exact solution and its derivative
`N`	number of intervals
`k`	degree of FE

Output

`E0`	error in L^2 for the solution
`E1`	error in L^2 for the derivative

Exercise 1

Consider the differential equation

$$-u'' = f \quad x \in (0,1) \qquad u(0) = u(1) = 0$$

where f is one of the following functions:

$$f_1(x) = 2 \qquad f_2(x) = -12x^2 + 12x - 2$$
$$f_3(x) = 4\pi^2 \sin(2\pi x) \quad f_4(x) = e^x(1 + x)$$

Solve the differential equation with P1 and P2 FEMs by means of the function `fem1d` for `N=[10 20 40 80 160 320]`.
Compute the L^2 relative errors of both the solution and its derivative by means of the function `errore`.
Compute the maximum error at the nodes.
Display the error in a loglog plot.

Solutions of exercise 1

The exact solution of exercise 1 has the following analytic expression:

$$u_1(x) = x(1-x)$$
$$u_2(x) = x^2(1-x)^2$$
$$u_3(x) = \sin(2\pi x)$$
$$u_4(x) = (e^x - 1)(1-x)$$

Esercizio 2

Consider the differential equation

$$-u'' + u = f \quad x \in (0,1) \qquad u(0) = u(1) = 0$$

where f is obtained so that the exact solution is the same as that of exercise 1.
Solve the differential equation with P1 and P2 FEMs using the function `fem1d` for `N=[10 20 40 80 160 320]`.
Compute the L^2 relative errors of both the solution and its derivative by means of the function `errore`.
Compute the maximum error at the nodes.
Display the error in a loglog plot.

Exercise 3

Solve the following differential equation

$$-u''(x) = f(x) \quad x \in (-1, 1), \qquad u(-1) = u(1) = 0$$

with $f(x) = \alpha(\alpha - 1)|x|^{\alpha-2}$.
Exact solution: $u(x) = 1 - |x|^{\alpha}$.
Use the following values $\alpha = 3, 2, 5/3, 3/2, 5/4$.
Plot the discrete solution together with the exact one.
Display the errors in the L^2-norm for the solution and its derivative and compute the convergence rate.

Singular perturbation

$$-\varepsilon u'' + u = 1 \quad x \in [0,1] \qquad u(0) = u(1) = 0$$

Solution:

$$u = \frac{-\sinh\left(\frac{x}{\sqrt{\varepsilon}}\right) + \sinh\left(\frac{x-1}{\sqrt{\varepsilon}}\right)}{\sinh\left(\frac{1}{\sqrt{\varepsilon}}\right)} + 1$$

- Take $\varepsilon =$1e-1,1e-3,1e-5, compute the solution for $N = 10$ and plot it together the exact solution.
- For $\varepsilon =$1e-3,1e-5 unexpected oscillations appear. Find the smallest N such that the numerical solution does not show oscillation.
- The elements of the stiffness matrix in this case are given by:

$$A_{ii} = \frac{2\varepsilon}{h} + \frac{2h}{3}, \quad A_{i\,i-1} = A_{i\,i+1} = -\frac{\varepsilon}{h} + \frac{h}{6}$$

Verify that oscillations appear as long as $A_{i\,i-1} = A_{i\,i+1} > 0$.