Finite element method in one dimension

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Outline

1. Problem setting
   - Weak formulation
   - Galerkin's method
2. Finite elements
   - Exercises

Advection diffusion problem

Problem in one dimension

Let

- $\Omega \subset \mathbb{R}^d$ bounded connected open set with Lipschitz boundary,
- $\mu : \Omega \to \mathbb{R}$, with $\mu \geq \mu_0 > 0$
- $\underline{\mathrm{b}} : \Omega \to \mathbb{R}^d$
- $\sigma : \Omega \to \mathbb{R}$ such that $0 \leq \sigma \leq \overline{\sigma}$
- $f : \Omega \to \mathbb{R}$

Find $u : \Omega \to \mathbb{R}$ such that

$$-\operatorname{div}(\mu \nabla u) + \underline{\mathrm{b}} \cdot \nabla u + \sigma u = f \quad \text{in } \Omega$$

boundary conditions of Dirichlet or Neumann type

Advection diffusion problem

Problem in one dimension

Let

- $\Omega =]a, b[$,
- $\mu \in \mathbb{R}$, with $\mu > 0$
- $\underline{\mathrm{b}} : \Omega \to \mathbb{R}$
- $\sigma : \Omega \to \mathbb{R}$ such that $0 \leq \sigma \leq \overline{\sigma}$
- $f : \Omega \to \mathbb{R}$

Find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{aligned} &-(\mu u'(x))' + \underline{\mathrm{b}} u'(x) + \sigma u(x) = f(x) \quad \text{for } x \in \Omega \\ &u(a) = \alpha, \quad u(b) = \beta. \end{aligned}$$

Problem with homogeneous Dirichlet boundary conditions

Problem

$$\begin{aligned} &-\mu u'' + \underline{b} u' + \sigma u = f \quad \text{in } \Omega =]a, b[\\ &u(a) = u(b) = 0 \end{aligned}$$

Functional setting

We introduce the following Hilbert spaces

$$\mathrm{L}^2(a,b) = \{v : (a,b) \to \mathbb{R} : \int_a^b v^2 \, dx < +\infty\}$$

$$H^1(a,b) = \{v \in \mathrm{L}^2(a,b) : v' \in \mathrm{L}^2(a,b)\}$$

$$V = H_0^1(a,b) = \{v \in H^1(a,b) : v(a) = v(b) = 0\}$$

endowed with the following norms

$$\|v\|_0 = \left(\int_a^b |v|^2 \, dx\right)^{1/2}, \quad \|v\|_1 = \left(\|v\|_0^2 + \|v'\|_0^2\right)^{1/2}$$

We multiply the equation by $v \in V$ (test function) and integrate on (a, b):

$$\int_a^b (-\mu u''(x) + \underline{b} u'(x) + \sigma u(x)) v(x)\, dx = \int_a^b f(x) v(x)\, dx$$

By integration by parts we have

$$\int_a^b \mu u''(x) v(x)\, dx = \left[\mu u'(x) v(x)\right]_a^b - \int_a^b \mu u'(x) v'(x)\, dx$$

from which

$$\int_a^b (\mu u'(x) v'(x) + \underline{b} u'(x) v(x) + \sigma u(x) v(x))\, dx = \int_a^b f(x) v(x)\, dx$$

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We set

$$a(u, v) = \int_a^b (\mu u'(x) v'(x) + \underline{b} u'(x) v(x) + \sigma u(x) v(x))\, dx,$$

$$F(v) = \int_a^b f(x) v(x)\, dx$$

Weak Problem

Find $u \in V$ such that

$$a(u, v) = F(v) \quad \forall v \in V.$$

Lax-Milgram Lemma

Assume that the bilinear form a is continuous and coercive and that the linear functional F is bounded, that is

$$\begin{aligned}
&\exists M_a > 0 \text{ s.t. } a(u, v) \leq M_a \|u\|_V \|v\|_V \\
&\exists \alpha > 0 \text{ s.t. } a(u, u) \geq \alpha \|u\|_V^2 \\
&F(v) \leq \|F\|_{V'} \|v\|_V
\end{aligned}$$

Then there exists a unique solution $u \in V$ of the weak problem. Moreover, the following a priori estimate holds true

$$\|u\|_V \leq \frac{\|F\|_{V'}}{\alpha}.$$

Galerkin's method

We consider a finite dimensional subspace V_h of V.

$$V_h \subset V, \qquad \dim V_h = N(h) < +\infty$$

Discrete problem

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Galerkin's method

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Let $\{\varphi_1, \ldots, \varphi_{N(h)}\}$ be a basis for V_h, hence $u_h = \sum_{j=1}^{N(h)} u_j \varphi_j$.

The vector $\underline{\mathrm{u}} \in \mathbb{R}^{N(h)}$ with components u_j satisfies

$$a\left(\sum_{j=1}^{N(h)} u_j \varphi_j, \varphi_i\right) = F(\varphi_i) \text{ for all } i = 1, \ldots, N(h).$$

Galerkin's method

Since a is bilinear we have

$$\sum_{j=1}^{N(h)} u_j a(\varphi_j, \varphi_i) = F(\varphi_i) \quad i = 1, \ldots, N(h).$$

For simplicity, let us assume that $\underline{\mathrm{b}}$ and σ are constant functions and define the following matrices with elements

$$K_{ij} = \int_a^b \varphi_j' \varphi_i' \, dx \qquad B_{ij} = \int_a^b \varphi_j' \varphi_i \, dx \qquad M_{ij} = \int_a^b \varphi_j \varphi_i \, dx,$$

and the vector $\underline{\mathrm{F}}$ with components

$$\underline{\mathrm{F}}_i = \int_a^b f \varphi_i \, dx.$$

Setting $A = \mu K + \underline{\mathrm{b}} B + \sigma M$, the discrete problem is equivalent to the following **linear system**

$$A \underline{\mathrm{u}} = \underline{\mathrm{F}}$$

with A not symmetric.

Error estimates

We recall the definition of the norms

$$\|v\|_0 = \left(\int_a^b v^2\,dx\right)^{1/2}, \quad \|v\|_V = \left(\|v\|_0^2 + \|v'\|_0^2\right)^{1/2}, \ \forall v \in V$$

Céa's Lemma

$$\|u - u_h\|_V \leq \frac{M}{\alpha} \inf_{v \in V_h} \|u - v_h\|_V.$$

The error is bounded by the best approximation in V_h.
NB The norm does contain information on $\underline{b}$.

Assembling the matrices

The matrix associated to the discretization of the first order term has the following structure:

$$B = \frac{1}{2}\begin{bmatrix} 0 & 1 & \dots & \dots & \dots & \dots & 0 \\ -1 & 0 & 1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -1 & 0 & 1 \\ 0 & \dots & \dots & \dots & \dots & -1 & 0 \end{bmatrix}$$

Function adv_diff

```
[x,u]=adv_diff(mu,f,bi,sigma,a,b,alfa,beta,N,k)
```

Input

`mu`	diffusion coefficient
`bi`	advection coefficient
`sigma`	function in the first order term u
`f`	right hand side
`a,b`	end points
`alfa, beta`	boundary values
`N`	number of the intervals
`k`	degree 1 or 2

Output

`x`	grid points
`u`	solution

Exercise 1

Let us consider the differential equation:

$$
\begin{aligned}
-u''(x) - 2u'(x) + e^x\, u(x) \\
= \sin(x)(1+e^x) + \cos(x)(2e^{-x}-1) \\
\text{for } x \in [-\pi, \pi] \\
u(-\pi) = -e^{\pi}, \quad u(\pi) = -e^{-\pi}.
\end{aligned}
$$

Exact solution: $u(x) = \sin(x) + e^{-x}\cos(x)$.

Use the function `adv_diff` to compute the solution and plot the discrete solution together with the exact solution.

Compute the error and display it in a loglog plot.

Advection dominated problems

Exercise 2

Let us consider the differential equation:

$$\begin{aligned} &-\varepsilon u''(x) + u'(x) = 0 \quad \text{for } x \in [0,1] \\ &u(0) = 0, \quad u(1) = 1. \end{aligned}$$

Exact solution: $u(x) = \frac{\exp((x-1)/\varepsilon) - \exp(-1/\varepsilon))}{(1-\exp(-1/\varepsilon))}$.

Solve the equation with `N=[10 20 40 80 160 320 640]`.

Compare the behavior of the solution for the following values of the diffusion coefficient $\varepsilon = 0.1$, $\varepsilon = 0.01$ e $\varepsilon = 0.001$.

Compute the Péclet number $\mathbb{P}e = \frac{|\underline{b}|h}{2\mu}$ and show that the oscillations disappear if $\mathbb{P}e < 1$.

What happens if we take $\underline{b} = -1$?

Artificial diffusion

Solve the problem in Exercise 2, using $\mu(1 + \mathbb{P}e)$ instead of μ and plot the solution with that obtained without the addition of artificial diffusion.