# Convergence analysis for hyperbolic evolution problems in mixed form

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**Abstract**

In this paper we present a convergence analysis for the space discretization of hyperbolic evolution problems in mixed form. The results of [1] are extended to this situation, showing the relationships between the approximation of the underlying eigenvalue problem and the space discretization of the corresponding evolution problem. The theory is applied to the finite element approximations of the wave equation in mixed form and to the Maxwell's equations. Some numerical results confirm the theory and make clear the critical points.

*Key words:* mixed finite elements, hyperbolic partial differential equations, wave equation, Maxwell's equations
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## 1 Introduction

In this paper we investigate the relationships between space discretization of partial differential equations and the approximation properties of the corresponding evolution problems. In [1] these relationships have been studied in the case of parabolic equations and the theory has been applied, in particular, to the mixed form of heat equation and to Stokes problem.

Here we extend the theory to hyperbolic evolution problems. Our unified approach include, as possible applications, the wave equation in mixed form and Maxwell's system. We refer the interested reader to [2] and to [3], respectively, for the specific results on those models.

It is well-known that the solution of our model problems can be represented, by separation of variables, as a Fourier series of the eigensolutions of the underlying space operator (Laplace and Maxwell operator, respectively). The same remark applies to the finite element approximations. Hence, it is not difficult to see that any choice of finite element spaces, which provides a good scheme for the approximation of the eigenvalues, can be successfully applied to the space semidiscretization of the corresponding evolution equation.

In the case of Laplace operator, thanks to the compactness of the problem and to the conformity of the approximation, any standard Ritz–Galerkin finite element scheme provides optimal convergence for the eigenvalue problem. In [1] the approximation of parabolic problems in mixed form has been studied in detail. In this case, it is no longer true that a good scheme for the approximation of the source (steady) problem always provides a convergent approximation to the corresponding mixed eigenvalue problem. Indeed, this pathology has been reported, for instance, in [4] and analyzed in [5]. This behavior implies that, if we use a scheme which provides a convergent approximation for the source problem but presents spurious eigensolutions when applied to the eigenvalue problem, there exist suitable data that may trigger spurious eigenmodes in the approximation to the evolution problem.

In this paper we develop a convergence theory for the space semidiscretization of a hyperbolic system of partial differential equations. Our analysis relies on the construction of suitable projections from the continuous spaces to the discrete ones and on the approximation properties of such projections. This hypotheses are closely related to the so called commuting (de Rham) diagram property and to the good approximation of the corresponding eigenvalue problem, see [5] and [6].

The paper is organized as follows: in Section 2 we present our abstract setting and prove our main convergence results under suitable compatibility assumptions. In Section 3 we describe the application to our theory to the wave equation in mixed form and to Maxwell system. Finally, in Section 4 we study an alternative formulation of Maxwell system, which is obtained after elimination of the magnetic field. Numerical results, demonstrating the necessity of our assumptions, are presented in Sections 3 and 4.

## 2 Approximation of hyperbolic problems in mixed form

### *2.1 Problem setting*

Let us consider four Hilbert spaces $\Phi$, $H_\Phi$, $\Xi$, and $H_\Xi$ with the indentification $H_\Xi \simeq H'_\Xi$ and the inclusions $\Xi \subset H_\Xi \subset \Xi'$ and $\Phi \subset H_\Phi$. Given a bilinear and continuous form $b : \Phi \times \Xi \to \mathbb{R}$ and a set of data $f \in L^2(0,T;H_\Xi)$, $\psi_0 \in \Phi$, and $\chi_0 \in \Xi$, we study the following problem: for a.e. $t \in [0,T]$ find $\psi(t) \in \Phi$ and $\chi(t) \in \Xi$ such that

$$\begin{aligned} &(\partial_t\psi(t),\varphi)_{H_\Phi} - b(\varphi,\chi(t)) = 0 && \forall\varphi \in \Phi \\ &(\partial_t\chi(t),\xi)_{H_\Xi} + b(\psi(t),\xi) = (f(t),\xi)_{H_\Xi} && \forall\xi \in \Xi \\ &\psi(0) = \psi_0, \qquad \chi(0) = \chi_0. \end{aligned} \tag{1}$$

As we shall see in the next subsections, the standard wave equation admits a variational formulation of the form (1).

In what follows we assume that the following inf-sup condition holds true:

$$\sup_{\varphi\in\Phi} \frac{b(\varphi,\chi)}{\|\varphi\|_\Phi} \geq \alpha\|\chi\|_\Xi \qquad \forall\,\chi \in \Xi. \tag{2}$$

Using $\xi = \chi(t)$ and $\varphi = \psi(t)$ as test functions in (1), summing up the two equations, and integrating with respect to time, we obtain $(0 < t < T)$:

$$\frac{1}{2}(\|\psi(t)\|^2_{H_\Phi} + \|\chi(t)\|^2_{H_\Xi}) = \int_0^t (f(t),\chi(t))_{H_\Xi}\,dt + \frac{1}{2}(\|\psi_0\|^2_{H_\Phi} + \|\chi_0\|^2_{H_\Xi}). \tag{3}$$

By Hölder inequality, we get

$$\|\psi\|^2_{L^\infty(H_\Phi)} + \|\chi\|^2_{L^\infty(H_\Xi)} \leq C\left(\|f\|^2_{L^1(H_\Xi)} + \|\psi\|^2_{H_\Phi} + \|\chi\|^2_{H_\Xi}\right). \tag{4}$$

**Remark 2.1** *If we consider the identification of $H_\Phi$ with its dual space $H'_\Phi$ instead of $H_\Xi \simeq H'_\Xi$, we can exchange the role of $\Phi$ and $\Xi$. Thus, the following problem can be analyzed with the same techniques: given $g \in L^2(0,T;H_\Phi)$, find*

*$\psi(t) \in \Phi$ and $\chi(t) \in \Xi$ for a.e. $t \in [0,T]$ such that*

$$\begin{aligned}
&(\partial_t \psi(t), \varphi)_{H_\Phi} - b(\varphi, \chi(t)) = (g(t), \varphi)_{H_\Phi} && \forall \varphi \in \Phi \\
&(\partial_t \chi(t), \xi)_{H_\Xi} + b(\psi(t), \xi) = 0 && \forall \xi \in \Xi \\
&\psi(0) = \psi_0, \qquad \chi(0) = \chi_0.
\end{aligned} \tag{5}$$

*In the following section, we shall see that Maxwell equations can be written in this form.*

**Remark 2.2** *Let $\zeta(t) \in \Xi$ be such that $\partial_t \zeta(t) = \chi(t)$ for a.e. $t$, $0 < t < T$, and that $\zeta(0) = \zeta_0$, with $\zeta_0 \in \Xi$ verifying*

$$(\psi_0, \varphi)_{H_\Phi} - b(\varphi, \zeta_0) = 0 \quad \forall \varphi \in \Phi. \tag{6}$$

*Note that problem (6) is well-posed due to (2), hence given $\psi_0 \in \Phi$ there exists a unique $\zeta_0 \in \Xi$ satisfying (6) with $\|\zeta_0\|_\Xi \leq C \|\psi_0\|_\Phi$.*

*Substituting $\chi(t)$ with $\partial_t \zeta(t)$ in (1) and integrating in time the first equation, thanks to (6) we obtain:*

$$\begin{aligned}
&(\psi(t), \varphi)_{H_\Phi} - b(\varphi, \zeta(t)) = 0 && \forall \varphi \in \Phi \\
&(\partial_{tt} \zeta(t), \xi)_{H_\Xi} + b(\psi(t), \xi) = (f(t), \xi)_{H_\Xi} && \forall \xi \in \Xi \\
&\zeta(0) = \zeta_0, \qquad \partial_t \zeta(0) = \chi_0.
\end{aligned} \tag{7}$$

*The system (7) is an equivalent formulation for (1) provided that (2) holds.*

### *2.2 Discretization and convergence*

Let $\Phi_h \subset \Phi$ and $\Xi_h \subset \Xi$ be two families of finite element spaces. The semidiscrete system associated with (1) reads: for a.e. $t \in [0,T]$ find $\psi_h(t) \in \Phi_h$ and $\chi_h(t) \in \Xi_h$ such that

$$\begin{aligned}
&(\partial_t \psi_h(t), \varphi)_{H_\Phi} - b(\varphi, \chi_h(t)) = 0 && \forall \varphi \in \Phi_h \\
&(\partial_t \chi_h(t), \xi)_{H_\Xi} + b(\psi_h(t), \xi) = (f(t), \xi)_{H_\Xi} && \forall \xi \in \Xi_h \\
&\psi_h(0) = \psi_{0,h}, \qquad \chi_h(0) = \chi_{0,h},
\end{aligned} \tag{8}$$

where $\psi_{0,h} \in \Phi_h$ and $\chi_{0,h} \in \Xi_h$ denote suitable approximations of the initial conditions.

Let us formally introduce two operators $\Pi_h : \Phi \to \Phi_h$ and $P_h : \Xi \to \Xi_h$ as follows:

$$\begin{aligned}
&(\psi - \Pi_h \psi, \varphi)_{H_\Phi} - b(\varphi, \zeta - P_h \zeta) = 0 && \forall \varphi \in \Phi_h \\
&b(\psi - \Pi_h \psi, \xi) = 0 && \forall \xi \in \Xi_h.
\end{aligned} \tag{9}$$

We assume that problem (9) is uniquely solvable for any choice of $(\psi, \zeta) \in \Phi \times \Xi$ (this is usually related to suitable compatibility conditions for $\Phi_h$ and $\Xi_h$, like the inf-sup conditions, see [7]).

The following lemma states that the operators defined in (9) commute with the time differentiation.

**Lemma 2.3** *Let $\psi : [0, T] \to \Phi$ and $\zeta : [0, T] \to \Xi$ be differentiable functions such that $\partial_t \psi(t) \in \Phi$ and $\partial_t \zeta(t) \in \Xi$ a.e. in $]0, T[$. Then*

$$\partial_t \Pi_h \psi(t) = \Pi_h \partial_t \psi(t), \quad \partial_t P_h \zeta(t) = P_h \partial_t \zeta(t) \tag{10}$$

*for a. e. $t \in ]0, T[$.*

*Proof.* By definition of the operators $\Pi_h$ and $P_h$ we have that $\Pi_h \psi(t) \in \Phi_h$ and $P_h \zeta(t) \in \Xi_h$ satisfy

$$\begin{aligned} &(\psi(t) - \Pi_h \psi(t), \varphi)_{H_\Phi} - b(\varphi, \zeta(t) - P_h \zeta(t)) = 0 && \forall \varphi \in \Phi_h \\ &b(\psi(t) - \Pi_h \psi(t), \xi) = 0 && \forall \xi \in \Xi_h. \end{aligned}$$

Differentiating with respect to $t$ we get

$$\begin{aligned} &(\partial_t(\psi(t) - \Pi_h \psi(t)), \varphi)_{H_\Phi} - b(\varphi, \partial_t(\zeta(t) - P_h \zeta(t))) = 0 && \forall \varphi \in \Phi_h \\ &b(\partial_t(\psi(t) - \Pi_h \psi(t)), \xi) = 0 && \forall \xi \in \Xi_h. \end{aligned}$$

On the other hand, we can define $\Pi_h \partial_t \psi(t)$ and $P_h \partial_t \zeta(t)$ as

$$\begin{aligned} &(\partial_t \psi(t) - \Pi_h \partial_t \psi(t), \varphi)_{H_\Phi} - b(\varphi, \partial_t \zeta(t) - P_h \partial_t \zeta(t)) = 0 && \forall \varphi \in \Phi_h \\ &b(\partial_t \psi(t) - \Pi_h \partial_t \psi(t), \xi) = 0 && \forall \xi \in \Xi_h. \end{aligned}$$

From the uniqueness property of problem (9) we get the result. □

**Theorem 2.4** *The following estimate holds true:*

$$\begin{aligned} &\|\psi - \psi_h\|_{L^\infty(H_\Phi)} + \|\chi - \chi_h\|_{L^\infty(H_\Xi)} \leq \\ &\quad C\Big(\|\partial_t(\chi - P_h \chi)\|_{L^1(H_\Xi)} + \|\psi - \Pi_h \psi\|_{L^\infty(H_\Phi)} + \\ &\quad \|\chi - P_h \chi\|_{L^\infty(H_\Xi)} + \|\Pi_h \psi_0 - \psi_{0,h}\|_{H_\Phi} + \|P_h \chi_0 - \chi_{0,h}\|_{H_\Xi}\Big). \end{aligned} \tag{11}$$

*Proof.* The error equations read:

$$\begin{aligned} &(\partial_t(\psi(t) - \psi_h(t)), \varphi) - b(\varphi, \chi(t) - \chi_h(t)) = 0 && \forall \varphi \in \Phi_h \\ &(\partial_t(\chi(t) - \chi_h(t)), \xi) + b(\psi(t) - \psi_h(t), \xi) = 0 && \forall \xi \in \Xi_h \end{aligned} \tag{12}$$

Let $(\psi(t), \chi(t)) \in \Phi \times \Xi$, a.e. $t \in (0, T)$, be the solution of (1); we define $\zeta(t) \in \Xi$, a.e. $t \in (0, T)$, according to Remark 2.2: i.e., $\zeta(t) \in \Xi$ is such that

$\partial_t\zeta = \chi$ and (6) holds. Then, we use (9) to define their projections $\Pi_h\psi(t)$ and $P_h\zeta(t)$ onto the discrete spaces. By (9), derivating in time the first equation, and using Lemma 2.3, we obtain:

$$\begin{aligned} &(\partial_t(\psi - \Pi_h\psi), \varphi)_{H_\Phi} - b(\varphi, \chi - P_h\chi) = 0 \quad && \forall \varphi \in \Phi_h \\ &b(\psi - \Pi_h\psi, \xi) = 0 && \forall \xi \in \Xi_h. \end{aligned} \tag{13}$$

We set $\varepsilon_h(t) = \Pi_h\psi(t) - \psi_h(t)$ and $e_h(t) = P_h\chi(t) - \chi_h(t)$, choose $\varphi = \varepsilon_h(t)$, $\xi = e_h(t)$, rearrange terms in (12), and obtain

$$\begin{aligned} &(\partial_t\varepsilon_h(t), \varepsilon_h(t)) - b(\varepsilon_h(t), e_h(t)) = \\ &\qquad -(\partial_t(\psi(t) - \Pi_h\psi(t)), \varepsilon_h(t)) + b(\varepsilon_h(t), \chi(t) - P_h\chi(t)) \\ &(\partial_t e_h(t), e_h(t)) + b(\varepsilon_h(t), e_h(t)) = \\ &\qquad -(\partial_t(\chi(t) - P_h\chi(t)), e_h(t)) - b(\psi(t) - \Pi_h\psi(t), e_h(t)). \end{aligned} \tag{14}$$

Using (13), and adding the two equations in (14), we obtain:

$$(\partial_t\varepsilon_h(h), \varepsilon_h(t)) + (\partial_t e_h(t), e_h(t)) = (\partial_t(\chi(t) - P_h\chi(t)), e_h(t))$$

By Cauchy-Schwartz and Hölder inequality, we have:

$$\begin{aligned} \|\varepsilon_h\|_{L^\infty(H_\Phi)} + \|e_h\|_{L^\infty(H_\Xi)} \le C\Big(&\|\partial_t(\chi - P_h\chi)\|_{L^1(H_\Xi)} + \\ &\|\Pi_h\psi_0 - \psi_{0,h}\|_{H_\Phi} + \|P_h\chi_0 - \chi_{0,h}\|_{H_\Xi}\Big). \end{aligned} \tag{15}$$

This implies the estimate (11) by triangle inequality. □

In order to use Theorem 2.4 for obtaining the convergence of the discrete solution of problem (8) to the continuous one (1), we make the following approximation assumptions: there exist $\omega_1(h)$ and $\omega_2(h)$ tending to zero as $h$ goes to zero such that

$$\begin{aligned} &\|\psi - \Pi_h\psi\|_{H_\Phi} \le \omega_1(h)\|\psi\|_{\Phi^+} \\ &\|\chi - P_h\chi\|_{H_\Xi} \le \omega_2(h)\|\chi\|_{\Xi^+} \end{aligned} \tag{16}$$

where $\Pi_h$ and $P_h$ are defined in (9) and $\Phi^+$ and $\Xi^+$ denote suitable subspaces of $\Phi$ and $\Xi$ containing the solution to (1) for a.e. $t$.

**Remark 2.5** *We explicitely observe that, in general, assumptions (16) are not straightforward consequences of standard error estimates for mixed problems. In particular, the operator $\Pi_h$ is a Fortin operator and the existence of $\omega_1(h)$ has been introduced for the first time in [5] as a necessary and sufficient condition for the good approximation of mixed eigenvalue problems (see also [8]).*

**Corollary 2.6** *Suppose that hypotheses (16) are satisfied and that the solution of (1) enjoys the additional regularity* $\partial_t \chi \in L^2(0,T;\Xi^+)$*. Then, taking* $\psi_{0,h} = \Pi_h \psi_0$ *and* $\chi_{0,h} = P_h \chi_0$*, the following estimate holds true:*

$$\|\psi - \psi_h\|_{L^\infty(H_\Phi)} + \|\chi - \chi_h\|_{L^\infty(H_\Xi)} \le C\left(\omega_1(h)\|\psi\|_{L^\infty(\Phi^+)} + \omega_2(h)\|\chi\|_{H^1(\Xi^+)}\right). \tag{17}$$

**Remark 2.7** *Note that any semi-discrete scheme for* (1) *results also in a semi-discrete scheme for* (7) *provided that for any* $\psi_{h,0} \in \Phi_h$*, there exists a* $\zeta_{h,0} \in \Xi_h$ *such that:*

$$(\psi_{h,0}, \varphi_h)_{H_\Phi} - b(\varphi_h, \zeta_{h,0}) = 0 \qquad \forall \varphi_h \in \Phi_h.$$

*As before, this condition is related to suitable compatibility conditions for* $\Phi_h$ *and* $\Xi_h$*, like the discrete counterpart of the inf-sup condition (2), see [7].*

*The corresponding error estimate reads*

$$\begin{aligned}\|\psi - \psi_h\|_{L^\infty(H_\Phi)} + &\|\partial_t(\zeta - \zeta_h)\|_{L^\infty(H_\Xi)} \le \\ &C\left(\omega_1(h)\|\psi\|_{L^\infty(\Phi^+)} + \omega_2(h)\|\partial_t \zeta\|_{H^1(\Xi^+)}\right).\end{aligned}$$

*Moreover, if the discrete inf-sup condition holds true, we have*

$$\begin{aligned}\|P_h\zeta(t) - \zeta_h(t)\|_\Xi \le C \sup_{\varphi \in \Phi_h} &\frac{b(\varphi, P_h\zeta(t) - \zeta_h(t))}{\|\varphi\|_\Phi} \le \\ C \sup_{\varphi \in \Phi_h} &\frac{(\psi(t) - \psi_h(t), \varphi)_{H_\Phi} + b(\varphi, P_h\zeta(t) - \zeta(t))}{\|\varphi\|_\Phi} \le \\ C\,&(\|\psi(t) - \psi_h(t)\|_{H_\Phi} + \|\psi(t) - \Pi_h\psi(t)\|_{H_\Phi}),\end{aligned}$$

*where we used (9) and the error equations. By triangle inequality we finally get*

$$\|\zeta - \zeta_h\|_{L^\infty(\Xi)} \le \|\zeta - P_h\zeta\|_{L^\infty(\Xi)} + C\left(\|\psi - \psi_h\|_{L^\infty(H_\Phi)} + \|\psi - \Pi_h\psi\|_{L^\infty(H_\Phi)}\right).$$

## 3 Applications

### *3.1 The wave equation*

Let $\Omega$ be a Lipschitz polygon (resp. polyhedron) in $\mathbb{R}^d$ ($d = 2$, resp $d = 3$). We consider the wave equation $\partial_t^2 u(t) - \Delta u(t) = f(t)$ with homogeneous boundary conditions and initial conditions $u(0) = u_0$, $\partial_t u(0) = u_1$; we observe that it fits within the framework of Section 2. Let us take $\Phi = H(\mathrm{div};\Omega)$,

$H_\Phi = L^2(\Omega)^d$, $\Xi = H_\Xi = L^2(\Omega)$ and define $b(\varphi, \xi) = -(\operatorname{div} \varphi, \xi)$. For the sake of simplicity, in order to write the mixed variational formulation, we now use the more standard notation $\boldsymbol{\sigma} = \psi$, $p = \chi$. With the identification $\boldsymbol{\sigma} = \mathbf{grad}\, u$ and $p = \partial_t u$, our problem reads: for a.e. $t \in [0, T]$ find $\boldsymbol{\sigma}(t) \in H(\operatorname{div}; \Omega)$ and $p(t) \in L^2(\Omega)$ such that

$$
\begin{aligned}
&(\partial_t \boldsymbol{\sigma}(t), \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, p(t)) = 0 && \forall \boldsymbol{\tau} \in H(\operatorname{div}; \Omega) \\
&(\partial_t p(t), q) - (\operatorname{div} \boldsymbol{\sigma}(t), q) = (f(t), q) && \forall q \in L^2(\Omega) \\
&\boldsymbol{\sigma}(0) = \mathbf{grad}\, u_0, \quad p(0) = u_1.
\end{aligned}
\tag{18}
$$

Denoting by $\Sigma_h$ and $Q_h$ finite element subspaces of $H(\operatorname{div}; \Omega)$ and $L^2(\Omega)$, respectively, the semidiscrete system associated to (18) reads: find $\boldsymbol{\sigma}_h(t) \in \Sigma_h$ and $p_h(t) \in Q_h$ such that

$$
\begin{aligned}
&(\partial_t \boldsymbol{\sigma}_h(t), \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, p_h(t)) = 0 && \forall \boldsymbol{\tau} \in \Sigma_h \\
&(\partial_t p_h(t), q) - (\operatorname{div} \boldsymbol{\sigma}_h(t), q) = (f(t), q) && \forall q \in Q_h \\
&\boldsymbol{\sigma}_h(0) = \boldsymbol{\sigma}_{0,h}, \quad p_h(0) = u_{1,h},
\end{aligned}
\tag{19}
$$

where $\boldsymbol{\sigma}_{0,h} \in \Sigma_h$ and $u_{1,h} \in Q_h$ are suitable approximations of the initial data.

**Remark 3.1** *The equivalent system* (7) *can be obtained by setting* $\psi = \boldsymbol{\sigma} = \mathbf{grad}\, u$ *and* $\zeta = u$. *The compatibility condition* (6) *is automatically fullfilled by integration by parts, recalling that* $u_0 = u(0) \in H_0^1(\Omega)$.

Following the same ideas of [1] we consider two possible choices of finite element spaces $\Sigma_h$ and $Q_h$ and study how they are related to the above analysis. Then, we report on two-dimensional numerical results which make use of formulation (19).

The first choice (RT) is based on the Raviart–Thomas elements introduced in [9,10], so that $\Sigma_h$ is the lowest order Raviart–Thomas space and $Q_h$ is the space of piecewise constants. It is well-known that this element satisfies the inf-sup conditions and our hypotheses (16) (see [7,11]). From Corollary 2.6 we have the optimal error estimate

$$
\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^\infty(L^2)} + \|p - p_h\|_{L^\infty(L^2)} \leq Ch \left( \|\boldsymbol{\sigma}\|_{L^\infty(H^1)} + \|p\|_{H^1(H^1)} \right).
$$

In [2] the use of Raviart–Thomas spaces for the approximation of the wave equation in mixed form had been already considered obtaining similar results with a different proof. There, a different construction of the operators $\Pi_h$ and $P_h$ had been proposed which required the inclusion $\operatorname{div} \Sigma_h \subset Q_h$. Our proof is more general and allows the use of elements which do not meet that inclusion. One example of such elements is given by the ABF family introduced in [12] for quadrilateral meshes.

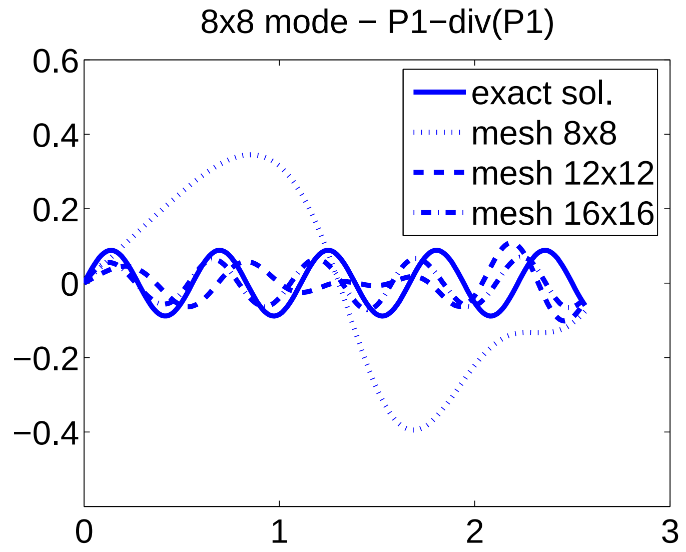


Fig. 1. The 8-by8 mode with the P1 method

The second choice (P1) takes $\Sigma_h$ as the space of continuous piecewise linear (componentwise) vectorfields and $Q_h = \operatorname{div} \Sigma_h$ which is made of piecewise constants. We consider a sequence of structured meshes; the square $\Omega$ is divided into uniform subsquares and each of them is divided into four triangles by its diagonals (criss-cross mesh). It is known that on this mesh the element P1 satisfies the inf-sup conditions (see [5]). On the other hand, the first estimate in (16) does not hold. Indeed, such estimate would imply that the (P1) scheme can be succesfully applied to the solution of the mixed Laplace eigenproblem and this is not the case (see [5] for more details).

The time discretization is performed by means of a suitably adapted Lax–Wendroff scheme. The computational domain is the square $\Omega = [0, \pi] \times [0, \pi]$. The solution to the wave equation with initial data $u_0(x, y) = 0$ and $u_1(x, y) = \sin ax \sin by$ is given by

$$u(x, y, t) = \frac{\sin\left(t\sqrt{a^2 + b^2}\right)}{\sqrt{a^2 + b^2}} \sin ax \sin by.$$

Our first numerical test considers the case $a = b = 8$ (we shall refer to this wave function as the 8-by-8 mode). In Figure 1 and 2 we plot our results for P1 and RT methods, respectively. In both figures, we plot the maximum values of the exact and the discrete solutions for different meshes.

It is clear that the RT method behaves better. In particular we can notice that on the 8-by-8 mesh the P1 solution definitely fails to capture the exact frequency and magnitude of the traveling wave. This behavior is strictly related to the presence of spurious eigensolution in the solution of Laplace eigenproblem with the P1 scheme (see [5,1] for more details). Indeed, the 8-by-8 mode

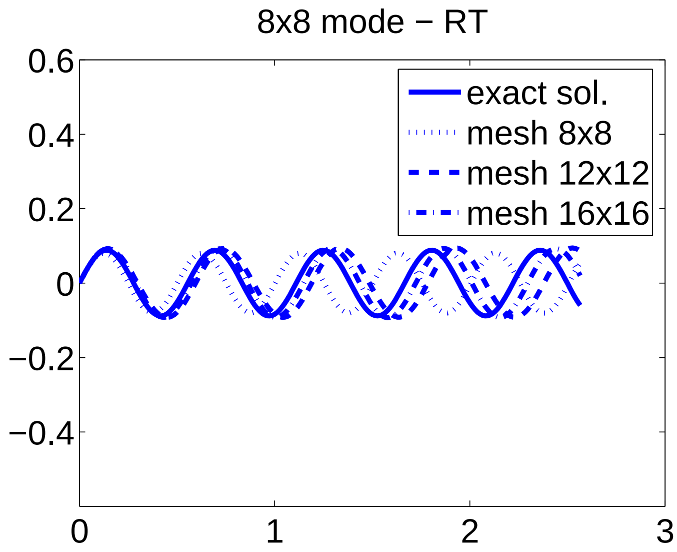


Fig. 2. The 8-by8 mode with the RT method

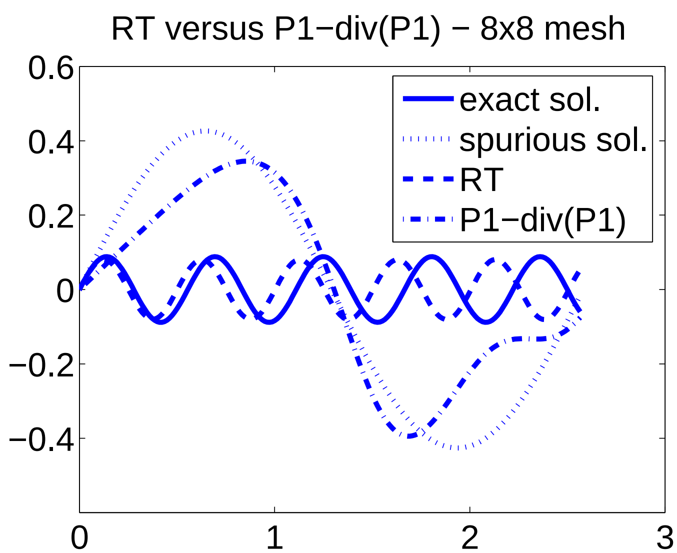


Fig. 3. The 8-by-8 mode on the 8-by-8 mesh

is very close to the first spurious eigenfunction computed with the P1 scheme on a 8-by-8 criss-cross mesh.

For this reason, we now investigate in a deeper way the different behavior of the two schemes when $u_0$ is the $N$-by-$N$ mode on a $N$-by-$N$ mesh. Figures 3, 4, and 5 show the results of these computations for $N$ equal to 8, 12, and 16. In these plots, we also report the discrete solution computed with the P1 scheme and having the first spurious eigenfunction on the same mesh as initial datum $u_0$.

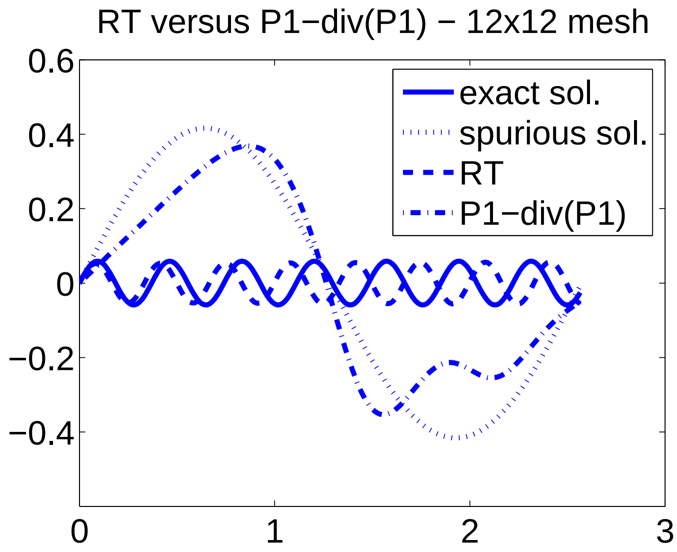


Fig. 4. The 12-by-12 mode on the 12-by-12 mesh

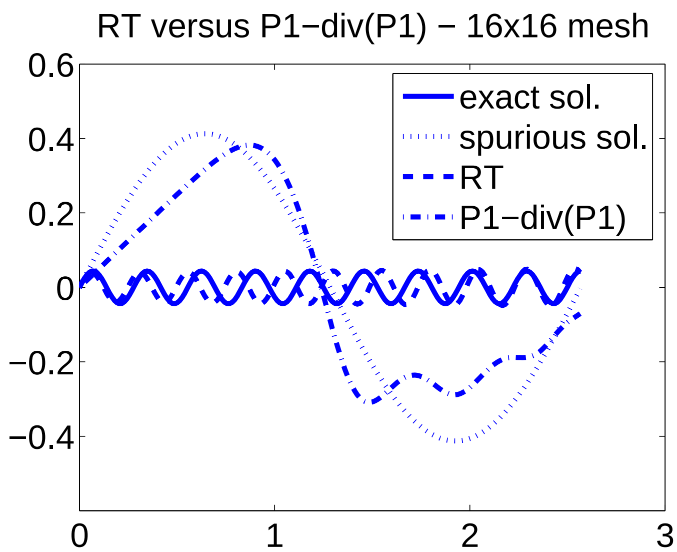


Fig. 5. The 16-by-16 mode on the 16-by-16 mesh

Then, in Figures 6 and 7, we plot the discrete solution $u_h(t)$ (recovered from $p_h(t)$ after integration) computed with the P1 and the RT scheme, respectively. The time $t$ is chosen in such a way that the norm of $u_h$ reaches its maximum value.

Finally, for the sake of comparison, in Figure 8 we plot the discrete solutions corresponding to the spurious eigenfunction as initial condition, computed with both the P1 and the RT scheme.

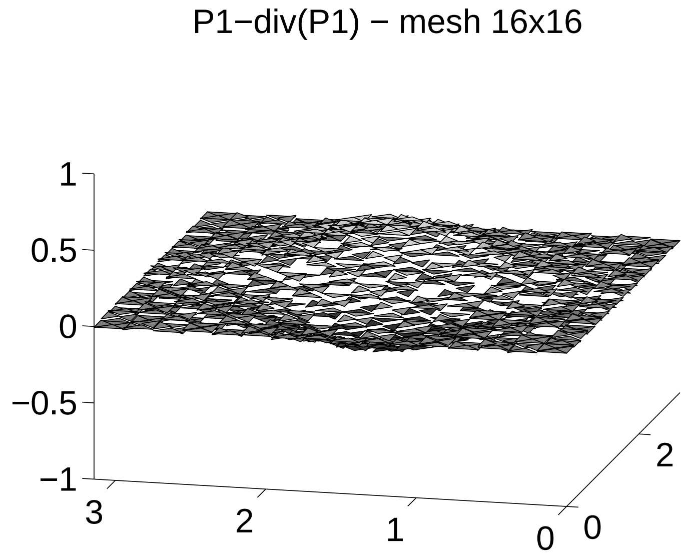


Fig. 6. The 16-by-16 mode on the 16-by-16 mesh with the P1 scheme

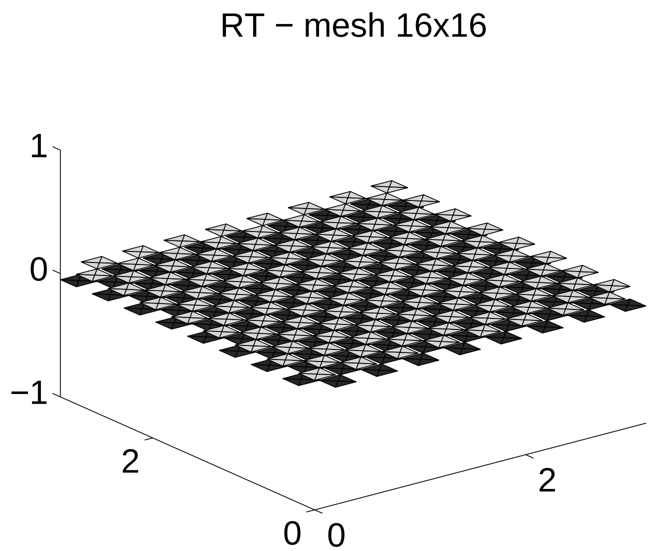


Fig. 7. The 16-by-16 mode on the 16-by-16 mesh with the RT scheme

### *3.2 Maxwell equations*

Let $\Omega \subseteq \mathbb{R}^d$ (d=2,3) be an open Lipschitz bounded polygon or polyhedron with outward unit vector $\mathbf{n}$. Let us denote by $\mathbf{E}$ and $\mathbf{H}$ the electric and the magnetic fields, respectively, and consider the Maxwell's system in the following form

$$\varepsilon \partial_t \mathbf{E} - \mathbf{curl}\, \mathbf{H} = \mathbf{J}, \qquad \mu \partial_t \mathbf{H} + \mathbf{curl}\, \mathbf{E} = 0, \tag{20}$$

where $\varepsilon$ and $\mu$, are the electric permittivity and magnetic permeability, respectively. The known function $\mathbf{J}$ specifies the applied current. We assume perfect

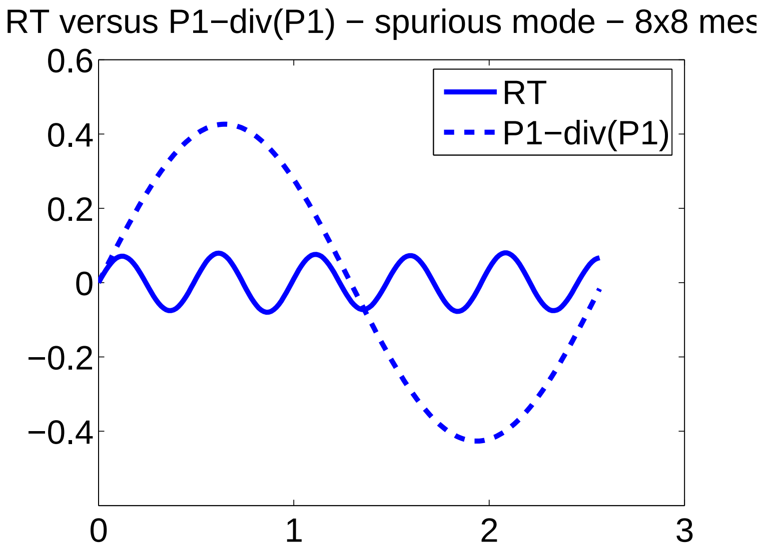


Fig. 8. Discrete solution corresponding to the spurious eigenfunction

conducting boundary condition on $\Omega$ so that

$$\mathbf{n} \times \mathbf{E} = 0 \qquad \text{on } \partial\Omega \times [0, T[. \tag{21}$$

In addition, we impose the following initial conditions

$$\mathbf{E}(0) = \mathbf{E}_0 \quad \text{in } \Omega, \qquad \mathbf{H}(0) = \mathbf{H}_0 \quad \text{in } \Omega, \tag{22}$$

with

$$\operatorname{div}(\varepsilon \mathbf{E}_0) = 0 \quad \text{in } \Omega, \qquad \operatorname{div}(\mu \mathbf{H}_0) = 0 \quad \text{in } \Omega, \ \mathbf{H}_0 \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega. \tag{23}$$

For $0 < t < T$ we assume also

$$\operatorname{div} \mathbf{J}(t) = 0 \quad \text{in } \Omega. \tag{24}$$

Conditions (23) and (24) together with (20) and (21) imply that for almost every $t \in ]0, T[$

$$\operatorname{div}(\varepsilon \mathbf{E}(t)) = 0 \quad \text{in } \Omega, \qquad \operatorname{div}(\mu \mathbf{H}(t)) = 0 \quad \text{in } \Omega, \ \mathbf{H}(t) \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

Moreover we assume $\varepsilon$ and $\mu$ to be real scalar functions independent of $t$ and satisfying a.e. in $\Omega$

$$0 < \varepsilon_* < \varepsilon < \varepsilon^* \qquad 0 < \mu_* < \mu < \mu^*. \tag{25}$$

The following functional spaces will be useful in order to write a variational formulation of problem (20):

$$
\begin{aligned}
&H_0(\mathbf{curl};\Omega) = \{\mathbf{v} \in L^2(\Omega)^d : \mathbf{curl}\,\mathbf{v} \in L^2(\Omega)^{2d-3},\ \mathbf{v}\times\mathbf{n} = 0 \text{ on } \partial\Omega\}\\
&H(\mathrm{div}^0;\Omega) = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathrm{div}\,\mathbf{v} = 0\}\\
&H(\mathrm{div}^0,\Omega;\mu) = \{\mathbf{v} \in H(\mathrm{div};\Omega) : \mathrm{div}(\mu\mathbf{v}) = 0\}\\
&H_0(\mathrm{div}^0;\Omega;\mu) = \{\mathbf{v} \in H(\mathrm{div}^0,\Omega;\mu) : (\mu\mathbf{v})\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}.
\end{aligned}
\tag{26}
$$

We multiply the first equation in (20) by $\boldsymbol{\varphi} \in H_0(\mathbf{curl};\Omega)$ and the second one by $\mu^{-1/2}\boldsymbol{\xi} \in H_0(\mathrm{div}^0;\Omega;\mu)$, integrate by part the second equation and use the boundary condition (21) This gives the following weak formulation of (20): given $\mathbf{J} \in L^2(0,T;H(\mathrm{div}^0;\Omega))$, for a.e. $t \in [0,T]$, find $\mathbf{E}(t) \in H_0(\mathbf{curl};\Omega)$ and $\mathbf{H}(t) \in H_0(\mathrm{div}^0;\Omega;\mu)$ such that

$$
\begin{aligned}
&(\varepsilon\partial_t\mathbf{E}(t),\boldsymbol{\varphi}) - (\mathbf{curl}\,\varphi,\mathbf{H}(t)) = (\mathbf{J}(t),\boldsymbol{\varphi}) \quad \forall\boldsymbol{\varphi} \in H_0(\mathbf{curl};\Omega)\\
&(\mu^{1/2}\partial_t\mathbf{H}(t),\boldsymbol{\xi}) + (\mu^{-1/2}\,\mathbf{curl}\,\mathbf{E}(t),\boldsymbol{\xi}) = 0 \quad \forall\boldsymbol{\xi} \in H_0(\mathrm{div}^0;\Omega;\mu^{1/2})\\
&\mathbf{E}(0) = \mathbf{E}_0 \qquad \mathbf{H}(0) = \mathbf{H}_0.
\end{aligned}
\tag{27}
$$

In order to cast problem (27) in the general form (1), we perform the change of variable $\mathbf{H}(t) = \mu^{-1/2}\mathbf{F}(t)$. Then we consider the following symmetric form of (27): for a.e. $t \in [0,T]$, find $\mathbf{E}(t) \in H_0(\mathbf{curl};\Omega)$ and $\mathbf{F}(t) \in H_0(\mathrm{div}^0;\Omega;\mu^{1/2})$ such that

$$
\begin{aligned}
&(\varepsilon\partial_t\mathbf{E}(t),\boldsymbol{\varphi}) - (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\varphi},\mathbf{F}(t)) = (\mathbf{J}(t),\boldsymbol{\varphi}) \quad \forall\boldsymbol{\varphi} \in H_0(\mathbf{curl};\Omega)\\
&(\partial_t\mathbf{F}(t),\boldsymbol{\xi}) + (\mu^{-1/2}\,\mathbf{curl}\,\mathbf{E}(t),\boldsymbol{\xi}) = 0 \quad \forall\boldsymbol{\xi} \in H_0(\mathrm{div}^0;\Omega;\mu^{1/2})\\
&\mathbf{E}(0) = \mathbf{E}_0 \qquad \mathbf{F}(0) = \mathbf{F}_0,
\end{aligned}
\tag{28}
$$

where $\mathbf{F}_0 = \mu^{1/2}\mathbf{H}_0$.

Problem (28) fits in the framework (5) by choosing $\Phi = H_0(\mathbf{curl};\Omega)$, $H_\Phi = H_\Xi = L^2(\Omega)^3$, $\Xi = H_0(\mathrm{div}^0;\Omega;\mu^{1/2})$, and $b(\boldsymbol{\varphi},\boldsymbol{\xi}) = (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\varphi},\boldsymbol{\xi})$.

Let us introduce two sequences $\mathcal{E}_h$ and $\mathcal{F}_h$ of finite element subspaces of $H_0(\mathbf{curl};\Omega)$ and $H_0(\mathrm{div}^0;\Omega;\mu^{1/2})$, respectively, then the semidiscrete system associated with (28) can be written as follows: for a.e. $t \in [0,T]$ find $\mathbf{E}_h(t) \in \mathcal{E}_h$ and $\mathbf{F}_h(t) \in \mathcal{F}_h$ such that

$$
\begin{aligned}
&(\varepsilon\partial_t\mathbf{E}_h(t),\boldsymbol{\varphi}) - (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\varphi},\mathbf{F}_h(t)) = (\mathbf{J}(t),\boldsymbol{\varphi}) \quad \forall\boldsymbol{\varphi} \in \mathcal{E}_h\\
&(\partial_t\mathbf{F}_h(t),\boldsymbol{\xi}) + (\mu^{-1/2}\,\mathbf{curl}\,\mathbf{E}_h(t),\boldsymbol{\xi}) = 0 \quad \forall\boldsymbol{\xi} \in \mathcal{F}_h\\
&\mathbf{E}(0) = \mathbf{E}_{0,h} \qquad \mathbf{F}(0) = \mathbf{F}_{0,h}.
\end{aligned}
\tag{29}
$$

where $\mathbf{E}_{0,h} \in \mathcal{E}_h$ and $\mathbf{F}_{0,h} \in \mathcal{F}_h$ are suitable approximations of $\mathbf{E}_0$ and $\mathbf{F}_0$, respectively.

In order to apply the theorems of Section 2 we define two operators $\Pi_h : H_0(\mathbf{curl};\Omega) \to \mathcal{E}_h$ and $P_h : H_0(\mathrm{div}^0;\Omega;\mu^{1/2}) \to \mathcal{F}_h$ satisfying (9) and (16), that is:

$$\begin{aligned} &(\varepsilon(\mathbf{E}-\Pi_h\mathbf{E}),\boldsymbol{\varphi})+(\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\varphi},\mathbf{F}-P_h\mathbf{F})=0 \quad && \forall\boldsymbol{\varphi}\in\mathcal{E}_h \\ &(\mu^{-1/2}\,\mathbf{curl}(\mathbf{E}-\Pi_h\mathbf{E}),\boldsymbol{\xi})=0 && \forall\boldsymbol{\xi}\in\mathcal{F}_h \end{aligned} \tag{30}$$

In [13,6], it has been proved that the well-posedness of (30) with the convergence property (16) is consequence of the so called *commuting diagram property* or *de Rham complex property*. There are basically two known families of finite element spaces that meet the commuting diagram property: the Raviart–Nédélec–Thomas one [10,9] (first and second kind on tetrahedra and the first kind on hexahedra) known as edge-face element family, and the one introduced more recently by Demkowicz and Vardapetyan [14,15].

For example, if $\mathcal{E}_h$ is the space of edge elements of order $k$, choosing $\mathcal{F}_h = \mu^{-1/2}\left(\mathcal{H}_h\cap H_0(\mathrm{div}^0;\Omega)\right)$, $\mathcal{H}_h$ being the corresponding space of face elements of order $k$ (see, e.g., [16]), we have $\mathcal{F}_h=\mu^{-1/2}\,\mathbf{curl}(\mathcal{E}_h)$ and our theory can be applied. In this case, using the estimates of [13] for the operators defined in (30) and standard interpolation estimates for edge and face elements (see, for instance, [17,18]) we get the following error estimate ($1/2<s\le k+1$)

$$\begin{aligned} &\|\mathbf{E}-\mathbf{E}_h\|_{L^\infty(L^2)}+\|\mathbf{F}-\mathbf{F}_h\|_{L^\infty(L^2)}\le \\ &\qquad Ch^s\left(\|\mathbf{E}\|_{L^\infty(H^s)}+\|\,\mathbf{curl}\,\mathbf{E}\|_{L^\infty(H^s)}+\|\mathbf{F}\|_{H^1(H^s)}\right) \end{aligned} \tag{31}$$

Estimates (31) are similar to the ones obtained in [3] for constant coefficients $\varepsilon$ and $\mu$. In [3] a different choice for the space $\mathcal{F}_h$ is also proposed which makes use of discontinuous piecewise polynomials instead of face elements (in particular, the divergence constraint is not present in the space). Our analysis does not allow this choice directly; indeed, equation (30) would not be uniquely solvable in this case. It would however be possible to filter out the gradient component in the definition of $P_h$, so that the rest of the analysis can be carried on in a slightly different way.

## 4 Alternative mixed formulation of Maxwell's equations as second order hyperbolic problem

A different approach for the approximation of (20) is to derive a second order hyperbolic problem for $\mathbf{E}$ by eliminating the magnetic field and taking into account the standard constitutive equations for linear media.

Given $\mathbf{E}_0$, $\mathbf{E}_1$ and $\mathbf{f}(t)$ for $t \in ]0,T[$, find $\mathbf{E}(t)$ such that, for $t \in ]0,T[$

$$\begin{array}{ll} \varepsilon\partial_{tt}\mathbf{E}(t) + \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}(t)) = \mathbf{f}(t) & \text{in } \Omega \times ]0,T[ \\ \operatorname{div}(\varepsilon\mathbf{E}(t)) = 0 & \text{in } \Omega \times ]0,T[ \\ \mathbf{E}(t) \times \mathbf{n} = 0 & \text{on } \partial\Omega \times ]0,T[ \\ \mathbf{E}(0) = \mathbf{E}_0, \qquad \partial_t\mathbf{E}(0) = \mathbf{E}_1 & \text{in } \Omega, \end{array} \tag{32}$$

where $\mathbf{f}(t) = \partial_t\mathbf{J}(t)$ and $\mathbf{E}_1 = \varepsilon^{-1}(\mathbf{J}(0) + \mathbf{curl}\,\mathbf{H}_0)$. Thanks to (23) and (24) we have also

$$\operatorname{div}(\varepsilon\mathbf{E}_i) = 0 \quad i = 1,2, \qquad \operatorname{div}(\mathbf{f}(t)) = 0 \quad t \in ]0,T[\,. \tag{33}$$

Then the divergence free constraint on $\mathbf{E}$ can be deduced from the first equation and the initial conditions in (32), hence we can drop it from the problem.

Given $\mathbf{f} : ]0,T[ \to H(\operatorname{div}^0;\Omega)$, $\mathbf{E}_0, \mathbf{E}_1, \in H_0(\mathbf{curl};\Omega)$, for almost every $t \in ]0,T[$, the variational formulation of problem (32) reads: find $\mathbf{u}(t) \in H_0(\mathbf{curl};\Omega)$ such that

$$\begin{array}{l} (\varepsilon\partial_{tt}\mathbf{u}(t),\mathbf{v}) + (\mu^{-1}\,\mathbf{curl}\,\mathbf{u}(t),\mathbf{curl}\,\mathbf{v}) = (\mathbf{f}(t),\mathbf{v}) \quad \forall\mathbf{v} \in H_0(\mathbf{curl};\Omega) \\ \mathbf{u}(0) = \mathbf{E}_0 \qquad \partial_t\mathbf{u}(0) = \mathbf{E}_1. \end{array} \tag{34}$$

In this section we do not transform (34) into a mixed form, but we want to highlight the relations between good approximation of (34) and that of an eigenproblem in mixed form.

Let $\lambda_i \in \mathbb{R}$ and $\mathbf{w}_i \in H_0(\mathbf{curl};\Omega) \cap H(\operatorname{div}^0;\Omega;\varepsilon)$, with $\mathbf{w}_i \neq 0$, be the eigenvalues and eigenvectors, respectively, of the Maxwell's operator that is for each $i$ they satisfy

$$(\mu^{-1}\,\mathbf{curl}\,\mathbf{w}_i,\mathbf{curl}\,\mathbf{v}) = \lambda_i(\mathbf{w}_i,\mathbf{v}) \quad \forall\mathbf{v} \in H_0(\mathbf{curl};\Omega) \cap H(\operatorname{div}^0;\Omega;\varepsilon). \tag{35}$$

Then we can represent the solution of problem (34) in terms of the following series:

$$\begin{aligned} \mathbf{u}(t) = \sum_{i=1}^{\infty} \Bigg[ & (\mathbf{E}_0,\mathbf{w}_i)\cos(\sqrt{\lambda_i}t) + \frac{1}{\sqrt{\lambda_i}}(\mathbf{E}_1,\mathbf{w}_i)\sin(\sqrt{\lambda_i}t) \\ & + \frac{1}{\sqrt{\lambda_i}}\int_0^t \sin\left(\sqrt{\lambda_i}(t-s)\right)(\mathbf{f}(s),\mathbf{w}_i)ds \Bigg]\mathbf{w}_i. \end{aligned} \tag{36}$$

Given a finite dimensional subspace $\mathbf{\Sigma}_h \subseteq H_0(\mathbf{curl};\Omega)$, the semidiscretization in space of (34) reads: given $\mathbf{f} : ]0,T[ \to H(\operatorname{div}^0;\Omega)$, $\mathbf{E}_0 \in H(\operatorname{div}^0;\Omega;\varepsilon)$ and $\mathbf{E}_0 \in H(\operatorname{div}^0;\Omega;\varepsilon)$, find $\mathbf{u}_h(t) \in \mathbf{\Sigma}_h$ such that, for almost every $t \in ]0,T[$

$$\begin{array}{l} \dfrac{d^2}{dt^2}(\varepsilon\mathbf{u}_h(t),\mathbf{v}) + (\mu^{-1}\,\mathbf{curl}\,\mathbf{u}_h(t),\mathbf{curl}\,\mathbf{v}) = (\mathbf{f}(t),\mathbf{v}) \quad \forall\mathbf{v} \in \mathbf{\Sigma}_h \\ \mathbf{u}_h(0) = \mathbf{E}_{0,h} \qquad \dfrac{\partial\mathbf{u}_h}{\partial t}(0) = \mathbf{E}_{1,h}, \end{array} \tag{37}$$

where $\mathbf{E}_{0,h} \in \mathbf{\Sigma}_h$ and $\mathbf{E}_{1,h} \in \mathbf{\Sigma}_h$ are suitable approximation of the initial data.

We can still represent the solution to (37) in a series similar to (36), using the approximation of the eigensolutions of (35). However, the numerical approximation of (35) presents some difficulties in dealing with the divergence free constraint, so that it is computationally convenient in the resolution of (35) to look for eigenfunctions $\mathbf{w}_i \in H_0(\mathbf{curl};\Omega)$. This implies, that a zero eigenvalue is added to the spectrum of the operator and that the associated eigenspace contains the gradients of all the functions in $H_0^1(\Omega)$.

Hence we consider the following discrete eigenproblem: find $\lambda_{ih} \in \mathbb{R}$ and $\mathbf{w}_{ih} \in \mathbf{\Sigma}_h$, with $\mathbf{w}_{ih} \neq 0$, such that for each $i$ they satisfy

$$(\mu^{-1}\,\mathbf{curl}\,\mathbf{w}_{ih}, \mathbf{curl}\,\mathbf{v}) = \lambda_{ih}(\mathbf{w}_{ih}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{\Sigma}_h. \tag{38}$$

Clearly, only the eigenvalues $\lambda_{ih} \neq 0$ are then considered as possible discretizations of eigenvalues of (35). In particular, a special care in the choice of the finite element space is needed in order that the spectrum of (38) is well separated into positive eigenvalues with (discrete) divergence free eigenfunctions and vanishing eigenvalues with (discretely) irrotational eigenfunctions (which have to be discarded); moreover, we have to make it sure that no spurious eigenmode polluting the spectrum is generated by the numerical scheme.

In [16], it is shown that a good finite element space for (38) is related to the good approximation of the following eigenproblem in mixed form: find $\lambda \in \mathbb{R}$ and $(\boldsymbol{\alpha}, \mathbf{p}) \in H_0(\mathbf{curl};\Omega) \times H_0(\mathrm{div}^0, \Omega; \mu^{1/2})$, with $(\boldsymbol{\alpha}, \mathbf{p}) \neq 0$, such that

$$\begin{aligned} &(\varepsilon\boldsymbol{\alpha}, \boldsymbol{\tau}) - (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{p}) = 0 && \forall \boldsymbol{\tau} \in H_0(\mathbf{curl};\Omega) \\ &(\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\alpha}, \mathbf{q}) = \lambda(\mathbf{p}, \mathbf{q}) && \forall \mathbf{q} \in H_0(\mathrm{div}^0, \Omega; \mu^{1/2}). \end{aligned} \tag{39}$$

Let us consider the following finite dimensional subspace $\mathbf{W}_h$ of $H_0(\mathrm{div}^0, \Omega; \mu^{1/2})$:

$$\mathbf{W}_h = \mu^{-1/2}\,\mathbf{curl}\,\mathbf{\Sigma}_h, \tag{40}$$

and the following discrete eigenproblem in mixed form: find $\lambda_h \in \mathbb{R}$ and $(\boldsymbol{\alpha}_h, \mathbf{p}_h) \in \mathbf{\Sigma}_h \times \mathbf{W}_h$, with $(\boldsymbol{\alpha}_h, \mathbf{p}_h) \neq 0$, such that

$$\begin{aligned} &(\varepsilon\boldsymbol{\alpha}_h, \boldsymbol{\tau}) - (\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{p}_h) = 0 && \forall \boldsymbol{\tau} \in \mathbf{\Sigma}_h \\ &(\mu^{-1/2}\,\mathbf{curl}\,\boldsymbol{\alpha}_h, \mathbf{q}) = \lambda_h(\mathbf{p}_h, \mathbf{q}) && \forall \mathbf{q} \in \mathbf{W}_h. \end{aligned} \tag{41}$$

In [16] it is also shown that if $\mathbf{\Sigma}_h$ is the space of edge elements of order $k$ and $\mathbf{W}_h = \mu^{-1/2}(\mathcal{H}_h \cap H(\mathrm{div}^0;\Omega))$, $\mathcal{H}_h$ being the space of face elements of order $k$, then the eigensolutions of (41) converge to those of (39). This implies that also the discrete solution to (37) converges to the continuous solution to (34) with the following representation:

$$\begin{aligned}\mathbf{u}_h(t) = \sum_{i=1}^{\infty} \Big[ (\mathbf{E}_0, \mathbf{w}_{i,h}) \cos(\sqrt{\lambda_{i,h}} t) + \frac{1}{\sqrt{\lambda_{i,h}}} (\mathbf{E}_1, \mathbf{w}_{i,h}) \sin(\sqrt{\lambda_{i,h}} t) \\ + \frac{1}{\sqrt{\lambda_{i,h}}} \int_0^t \sin\left(\sqrt{\lambda_{i,h}}(t-s)\right) (\mathbf{f}(s), \mathbf{w}_{i,h}) ds \Big] \mathbf{w}_{i,h}.\end{aligned} \tag{42}$$

**Remark 4.1** *It is worth noticing that the converge analysis of* (37) *with this particular choice of* $\mathbf{\Sigma}_h$ *is carried out in [20] and makes indirect use of the spectral theory developed in [16] and [13].*

In what follows we describe some numerical tests aiming at proving that, if (9) and (16) are not met by the underlying steady mixed formulations (39) and (41), then the solutions of (37) can present an undesired behavior.

Let then $\Omega$ be a square domain and $\mathbf{\Sigma}_h$ be the space of piecewise linear vectors on a criss-cross mesh. As in the case of the Laplace operator, this choice in (38) provides spurious eigenvalues, [16]. Let us consider the eigenfunction $\overline{\mathbf{w}}_h$ associated to the first spurious eigenvalue $\bar{\lambda}_h$ (which is a number close to six). As already pointed out, such eigenfunction has a clear checkerboard pattern, so that it should be associated to an high frequency mode. Let us now take the following data in (37)

$$\mathbf{E}_{0,h} = \overline{\mathbf{w}}_h, \qquad \mathbf{E}_{1,h} = 0, \qquad \mathbf{f} = 0. \tag{43}$$

Solving (37) with the P1 method on criss-cross mesh, and using the representation (42) we obtain the following solution:

$$\mathbf{u}_h(t) = \cos(\sqrt{\bar{\lambda}_h} t)\overline{\mathbf{w}}_h. \tag{44}$$

As explained above, in our case $\bar{\lambda}_h$ is a relatively small number which does not reflect the highly oscillatory behavior of $\overline{\mathbf{w}}_h$. This implies that the discrete solution (44) will not provide a good approximation to the true solution. In general, we are expecting the exact solution to present a much higher frequency (in time) than the discrete one.

Let us take, for our numerical computation, a mesh of $16 \times 16$ squares subdivided into 4 subsquares by their diagonals. For simplicity, we consider $\varepsilon = \mu = 1$. In [1] it has been shown that the discrete solution $\overline{\mathbf{w}}_h$ is strictly related to the eigenmode $(15, 15)$ of (35). We are then expecting an exact solution with a high time frequency ($15^2 + 15^2 = 450$), while the discrete solution, given by (44), has a frequency related to $\bar{\lambda}_h \simeq 6$. This bad behavior is clearly observed in Figure 9, where the values at a fixed arbitrary point of the first component of the exact solution associated to the eigenmode $(15, 15)$ and of the computed solution (44) are plotted on the same graph. In the same Figure (right) we

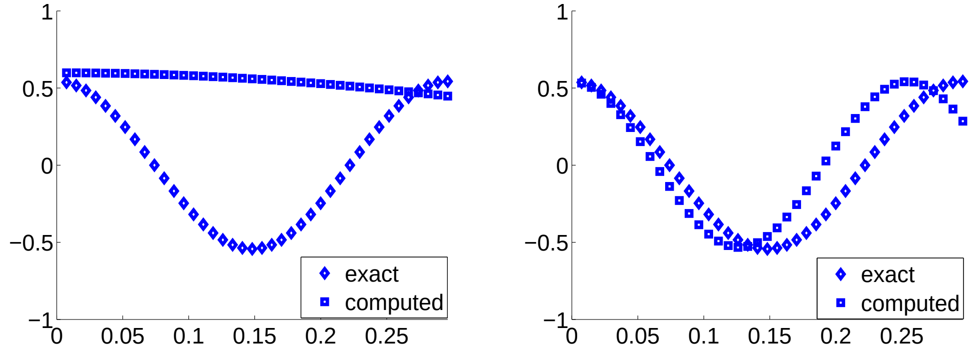


Fig. 9. Phase error in the computation of the evolution of a spurious eigenfunction (left) and in the computation of the interpolation of the exact eigenfunction $(15, 15)$ (right) which is closely related to the spurious eigenfunction

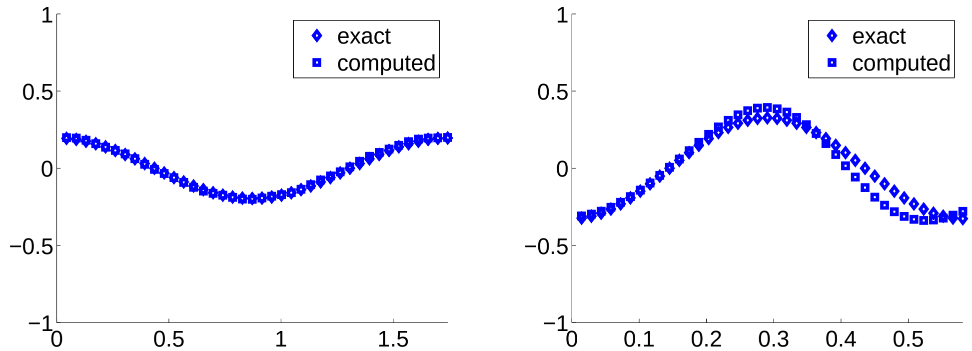


Fig. 10. Phase error in the computation of the evolution of mode $(3, 2)$ (left) and of mode $(6, 9)$ (right)

plot a similar graph, where the discrete solution is computed with an initial datum which is the interpolation of the exact eigenfunction $(15, 15)$. It may be noticed that, in this case, the phase error is much less evident and is fully justified by the relatively coarse mesh (a $16 \times 16$ mesh which is basically of the same size as the wavelength).

To confirm that the P1 method performs well when applied to a smooth datum and that the phase error is negligible when the mesh is fine compared to the wavelength, we plot in Figure 10 graphs similar to those of Figure 9 corresponding to the modes $(3, 2)$ and $(6, 9)$, respectively.

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