A finite element approach for the immersed boundary method

Daniele Boffi [a] and Lucia Gastaldi [b,1]

[a] *Dipartimento di Matematica, Università di Pavia, via Ferrata 1 Pavia, Italy*

[b] *Dipartimento di Matematica, Università di Brescia, via Valotti 9 Brescia, Italy*

Abstract

We consider a modification of the immersed boundary method which makes use of a finite element spatial discretization. We describe the method for a two-dimensional model problem and justify its variational formulation. We provide a preliminary analysis of the continuous problem in a one-dimensional setting using a fixed point theorem and a compactness argument. Finally, we report on some numerical tests which demonstrate the stability and robustness of the algorithm.

Key words: Immersed boundary method, finite element method

1 Introduction

The immersed boundary (IB) method was developed for the computer simulation of fluid-structure interaction, especially in biological fluid dynamics (see [7] and the bibliography quoted therein). The coupling of fluids and solids is the central feature in the study of the mechanics of the heart, arteries, veins, micro-circulation, and pulmonary blood flow. Nevertheless, the modeling of strong hemodynamic interaction with flexible structures is often limited by severe fluid mesh distortions around the structures experiencing large deformations and displacements. In particular, the computation requirement with respect to evolving or adaptive meshing becomes impractical in the modeling of deformable cells interacting with viscous fluid and flexible vessels. In the IB method, the structure is thought as a part of the fluid where additional forces

[1] Corresponding author. Address: Dipartimento di Matematica, Università di Brescia. Via Valotti 9, I-25133, Italy; Fax: +39 030 3716745; E-mail: gastaldi@ing.unibs.it

are applied, and where additional mass may be localized. Therefore, instead of separating the system in its two components coupled by boundary conditions, as is conventionally done, the incompressible Navier-Stokes equations, with a nonuniform mass density and an applied elastic force density, are used in order to describe the coupled motion of the hydro-elastic system in a unified way.

The IB method is at the same time a mathematical formulation and a numerical scheme. The mathematical formulation is based on the use of Eulerian variables to describe the dynamic of the fluid and of Lagrangian variables along the moving structure. The force exerted by the structure on the fluid is taken into account by means of a Dirac function constructed according to certain principles. The original IB method uses finite differences in order to approximate the Navier-Stokes equations. We aim on providing a suitable modification of the IB method which makes use of finite elements. For that reason we need a variational formulation of the problem. The first step in this direction is given by making precise the definition of the source term in the Navier-Stokes equations which involves a Dirac function and describes the elastic reaction of the immersed boundary. The problem is highly nonlinear and presents several difficulties related with the lacking of regularity of the solution of the Navier-Stokes equations due to such source term. Apparently, no rigorous analysis of the problem is available at the moment. We start with the analysis of the existence of the solution in a very simple one-dimensional case, using a fixed point argument.

The outline of the paper is the following: in the next section the formulation of the problem will be introduced with some considerations on the Dirac function which describes the applied elastic force density due to the presence of the structure. Moreover, the variational formulation of the problem is given. Sect. 3 is devoted to the time and space discretization. In Sect. 4 we discuss the existence of the solution for a one-dimensional problem. The last section contains some numerical investigations.

2 Formulation of the problem

We consider the model problem of a viscous incompressible fluid in a two-dimensional square domain Ω containing an immersed massless elastic boundary in the form of a curve. To be more precise, for all $t \in [0,T]$, let Γ_t be a simple closed curve, the configuration of which is given in a parametric form, $\mathbf{X}(s,t)$, $0 \le s \le L$, $\mathbf{X}(0,t) = \mathbf{X}(L,t)$ (see [8,4]). The equations of motion of the system are

$$\rho\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\,\mathbf{u} + \nabla\,p = \mathbf{F} \quad \text{in } \Omega\times\,]0,T[\tag{1}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times\,]0,T[\tag{2}$$

$$\mathbf{F}(\mathbf{x},t) = \int_0^L \mathbf{f}(s,t)\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds \quad \forall\mathbf{x}\in\Omega,\ t\in\,]0,T[\tag{3}$$

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s\in[0,L],\ t\in\,]0,T[\tag{4}$$

Here $\mathbf{u}$ is the fluid velocity and p is the fluid pressure. The coefficients ρ and μ are the constant fluid density and viscosity. Equations (1) and (2) are the usual incompressible Navier-Stokes equations. $\mathbf{F}$ is the force density generated by the boundary on the fluid. The force exerted by the element of boundary on the fluid is $\mathbf{f}$. The modelization of $\mathbf{f}$ varies from one problem to the other (see, e.g. [5,6,8,10,3]); for example, we can have $\mathbf{f}(s,t) = \mathbf{S}(\mathbf{X}(s,t),t)$ where $\mathbf{S}$ satisfies a generalized Hooke's law if the boundary is elastic. The function δ in the integrals is the two-dimensional Dirac function concentrated at the origin. Equation (4) is equivalent to the no-slip condition that the fluid sticks to the boundary.

The problem is completely described once we fix boundary and initial conditions:

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega\times\,]0,T[\tag{5}$$

$$\mathbf{u}(\cdot,0) = \mathbf{u}_0(\cdot) \qquad \text{on } \Omega \tag{6}$$

$$\mathbf{X}(s,0) = \mathbf{X}_0(s) \qquad \forall s\in[0,L]. \tag{7}$$

To start with we shall consider the linear evolutive Stokes equations, which can be obtained from equations (1), (2) by dropping the nonlinear term and a force density which does not depend on $\mathbf{X}$. The problem is now

$$\rho\frac{\partial \mathbf{u}}{\partial t} - \mu\Delta\mathbf{u} + \nabla\,p = \mathbf{F} \quad \text{in } \Omega\times\,]0,T[\tag{8}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega\times\,]0,T[\tag{9}$$

$$\mathbf{F}(\mathbf{x},t) = \int_0^L \mathbf{f}(s,t)\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds \quad \forall\mathbf{x}\in\Omega,\ t\in\,]0,T[\tag{10}$$

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s\in[0,L],\ t\in\,]0,T[\tag{11}$$

with the same initial and boundary conditions (5)- (7).

Notice that, the integral in equation (10) needs a more precise definition which will be given in the next Lemma. We observe that, in equation (10), the boundary force $\mathbf{f}$ is integrated with a two-dimensional Dirac function over the one-dimensional curve Γ_t, parametrically represented by X. Hence the force density $\mathbf{F}$ is a one-dimensional Dirac function along Γ_t.

Lemma 1 *Assume that, for all $t \in [0,T]$, the curve Γ_t is Lipschitz continuous. Then for all $t \in]0,T[$, the force density $\mathbf{F}(t)$, defined formally in (10), is a distribution function belonging to $H^{-1}(\Omega)^2$ defined as follows: for all $\mathbf{v} \in H_0^1(\Omega)^2$*

$$ {}_{H^{-1}}\!< \mathbf{F}(t), \mathbf{v} >_{H_0^1} = \int_0^L \mathbf{f}(s,t) \cdot \mathbf{v}(\mathbf{X}(s,t))\, ds \quad \forall t \in]0,T[\,. \tag{12} $$

Proof. Let us introduce the family of continuous positive functions δ_ε, with $\varepsilon > 0$, such that

$$ \mathrm{supp}(\delta_\varepsilon) \subseteq B_\varepsilon(0), \quad \int_{B_\varepsilon(0)} \delta_\varepsilon \mathrm{d}\mathbf{x} = 1, \quad \delta_\varepsilon \longrightarrow \delta \text{ in } \mathcal{D}', \tag{13} $$

where $B_\varepsilon(0)$ is the ball of radius ε centered at the origin, and define

$$ \mathbf{F}_\varepsilon(\mathbf{x},t) = \int_0^L \mathbf{f}(s,t)\delta_\varepsilon(\mathbf{x} - \mathbf{X}(s,t))\, ds \quad \forall \mathbf{x} \in \Omega,\ t \in]0,T[\,. $$

For all $t \in]0,T[$, the duality pairing of $\mathbf{F}_\varepsilon(t)$ with $\mathbf{w} \in \mathcal{D}(\Omega)^2$ can be written as an integral over Ω, as follows:

$$ \begin{aligned} {}_{\mathcal{D}'}\!< \mathbf{F}_\varepsilon(t), \mathbf{w} >_{\mathcal{D}} &= \int_\Omega \mathbf{F}_\varepsilon(\mathbf{x},t) \cdot \mathbf{w}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &= \int_\Omega (\int_0^L \mathbf{f}(s,t)\delta_\varepsilon(\mathbf{x} - \mathbf{X}(s,t))\, ds) \cdot \mathbf{w}(\mathbf{x})\mathrm{d}\mathbf{x} \\ &= \int_0^L \mathbf{f}(s,t) \left(\int_\Omega \delta_\varepsilon(\mathbf{x} - \mathbf{X}(s,t)) \cdot \mathbf{w}(\mathbf{x})\mathrm{d}\mathbf{x} \right) ds \end{aligned} \tag{14} $$

Passing to the limit as ε goes to zero in (14), we obtain thanks to (13)

$$ \begin{aligned} & \lim_{\varepsilon \to 0} {}_{\mathcal{D}'}\!< \mathbf{F}_\varepsilon(t), \mathbf{w} >_{\mathcal{D}} \\ &\quad = \lim_{\varepsilon \to 0} \int_0^L \mathbf{f}(s,t) \left(\int_\Omega \delta_\varepsilon(\mathbf{x} - \mathbf{X}(s,t)) \cdot \mathbf{w}(\mathbf{x})\mathrm{d}\mathbf{x} \right) ds \\ &\quad = \int_0^L \mathbf{f}(s,t) \left(\lim_{\varepsilon \to 0} \int_\Omega \delta_\varepsilon(\mathbf{x} - \mathbf{X}(s,t)) \cdot \mathbf{w}(\mathbf{x})\mathrm{d}\mathbf{x} \right) ds \\ &\quad = \int_0^L \mathbf{f}(s,t) {}_{\mathcal{D}'}\!< \delta_{\mathbf{X}(s,t)}, \mathbf{w} >_{\mathcal{D}}\ ds \\ &\quad = \int_0^L \mathbf{f}(s,t)\mathbf{w}(\mathbf{X}(s,t))\, ds. \end{aligned} \tag{15} $$

Here $\delta_{\mathbf{X}(s,t)}$ denotes the two-dimensional Dirac function concentrated at the point $\mathbf{X}(s,t)$.

We denote by $\mathbf{F}$ the limit in $\mathcal{D}'(\Omega)$ of $\mathbf{F}_\varepsilon$, as ε goes to zero, hence we get from (15)

$$ {}_{\mathcal{D}'}< \mathbf{F}(t), \mathbf{w} >_{\mathcal{D}} = \lim_{\varepsilon \to 0} {}_{\mathcal{D}'}< \mathbf{F}_\varepsilon(t), \mathbf{w} >_{\mathcal{D}} = \int_0^L \mathbf{f}(s,t)\mathbf{w}(\mathbf{X}(s,t))\, ds. \tag{16} $$

The integral in the right hand side is defined for all the functions $\mathbf{w}$, such that the trace $\mathbf{w}(\mathbf{X}(\cdot,t))$ along the curve under consideration belongs to $L^2([0,L])^2$. For all $t \in]0,T[$, $\mathbf{X}(\cdot,t) : [0,L] \to \Omega$ represents a Lipschitz curve in Ω, then the trace operator $\tau_\mathbf{X}$ associating to a function $\mathbf{w} : \Omega \to \mathbb{R}^2$ its restriction along Γ_t, is well-defined for $\mathbf{w} \in H_0^1(\Omega)^2$. Moreover one has that

$$ \tau_\mathbf{X} : H_0^1(\Omega)^2 \to H^{1/2}(\Gamma_t)^2. $$

Since $\mathcal{D}(\Omega)$ is dense in $H_0^1(\Omega)^2$, we can extend $\mathbf{F}(t)$ to a linear continuous functional over $H_0^1(\Omega)^2$ and (12) holds true. $\square$

In view of the finite element discretization of the problem, we introduce a variational formulation of equations (8)- (9). We recall that $L_0^2(\Omega)$ is the subset of $L^2(\Omega)$ containing the elements with zero mean value.

Problem 1 *Given* $\mathbf{f} \in L^2(]0,L[\times]0,T[)$, *for all* $t \in]0,T[$, *find* $(\mathbf{u}(t), p(t)) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ *and* $\mathbf{X} : [0,L]\times]0,T[\to \Omega$, *such that*

$$ \begin{aligned} &\frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + \mu(\boldsymbol{\nabla}\,\mathbf{u}(t), \boldsymbol{\nabla}\,\mathbf{v}) - (\nabla\cdot\mathbf{v}, p(t)) = < \mathbf{F}(t), \mathbf{v} > \ \forall \mathbf{v} \in H_0^1(\Omega)^2 \\ &(\nabla\cdot\mathbf{u}(t), q) = 0 \qquad\qquad\qquad\qquad \forall q \in L_0^2(\Omega) \end{aligned} \tag{17} $$

$$ < \mathbf{F}(t), \mathbf{v} > = \int_0^L \mathbf{f}(s,t)\mathbf{v}(\mathbf{X}(s,t))\, ds, \quad \forall \mathbf{v} \in H_0^1(\Omega)^2 \tag{18} $$

$$ \frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in [0,L], \tag{19} $$

$$ \mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \ \forall \mathbf{x} \in \Omega, \tag{20} $$

$$ \mathbf{X}(s,0) = \mathbf{X}_0(s) \ \forall s \in [0,L], \qquad \mathbf{X}(0,t) = \mathbf{X}(L,t), \ \forall t \in [0,T]. \tag{21} $$

In (17), $(\cdot,\cdot)$ stands for the usual scalar product in $L^2(\Omega)$, while $< \cdot,\cdot >$ denotes the duality pairing between $H^{-1}(\Omega)^2$ and $H_0^1(\Omega)^2$.

3 Finite element discretization

Let $\mathcal{T}_h$ be a triangulation of Ω. We denote by h_x the biggest diameter of the elements of $\mathcal{T}_h$. Let us consider two finite dimensional subspaces $\mathbf{V}_h \subseteq H_0^1(\Omega)^2$ and $Q_h \subseteq L_0^2(\Omega)$. Next, let s_i, $i = 0, \cdots, m$ with $s_0 = 0$ and $s_m = L$, be $m+1$

points of the interval $[0, L]$. We set $h_s = \max_{0 \le i \le m} |s_i - s_{i-1}|$. Let S_h be the finite element space of piecewise linear vectors defined on $[0, L]$ as follows

$$\begin{aligned}\mathbf{S}_h = \{\mathbf{Y} \in C^0([0, L]; \Omega) : \mathbf{Y}|_{[s_{i-1}, s_i]} \in \mathcal{P}^1([s_{i-1}, s_i])^2,\ i = 1, \cdots, m, \\ \mathbf{Y}(s_0) = \mathbf{Y}(s_m)\}\end{aligned} \tag{22}$$

where $\mathcal{P}^1(I)$ stands for the space of affine polynomials on the interval I. For an element $\mathbf{Y} \in \mathbf{S}_h$ we shall use also the following notation $\mathbf{Y}_i = \mathbf{Y}(s_i)$ for $i = 0, \cdots, m$.

Then the discretization in space of Problem 1 reads:

Problem 2 *Given* $\mathbf{f} \in L^2(]0, L[\times]0, T[)$, *for all* $t \in]0, T[$, *find* $(\mathbf{u}_h(t), p_h(t)) \in \mathbf{V}_h \times Q_h$ *and* $\mathbf{X}_h(t) \in \mathbf{S}_h$ *such that*

$$\begin{aligned}&\rho \frac{d}{dt}(\mathbf{u}_h(t), \mathbf{v}) + \mu(\boldsymbol{\nabla}\, \mathbf{u}_h(t), \boldsymbol{\nabla}\, \mathbf{v}) - (\nabla\cdot \mathbf{v}, p_h(t)) =< \mathbf{F}_h(t), \mathbf{v} > \forall \mathbf{v} \in \mathbf{V}_h \\ &(\nabla\cdot \mathbf{u}_h(t), q) = 0 \qquad\qquad \forall q \in Q_h\end{aligned} \tag{23}$$

$$< \mathbf{F}_h(t), \mathbf{v} >_h = \sum_{i=0}^{m} \omega_i \mathbf{f}(s_i, t)\mathbf{v}(\mathbf{X}_{hi}(t)), \quad \forall \mathbf{v} \in \mathbf{V}_h \tag{24}$$

$$\frac{d\mathbf{X}_{hi}}{dt}(t) = \mathbf{u}_h(\mathbf{X}_{hi}(t), t) \quad \forall i = 1, \cdots, m \tag{25}$$

$$\mathbf{u}_h(\mathbf{x}, 0) = \mathbf{u}_{0h}(\mathbf{x})\ \forall \mathbf{x} \in \Omega, \tag{26}$$

$$\mathbf{X}_{hi}(0) = \mathbf{X}_0(s_i)\ \forall i = 1, \cdots, m. \tag{27}$$

The sum in the right hand side of (24) represents a suitable quadrature formula with nodes s_i and weights ω_i. We observe that the mesh parameters h_x and h_s should be chosen properly so that the error due to the quadrature rule does not affect the accuracy of the solution.

It is well-known that the finite element spaces $\mathbf{V}_h$ and Q_h have to be chosen properly in order to garantee the existence, uniqueness, stability and convergence properties of the solution to (23), see, e.g. [1].

Let us now introduce a time discretization scheme of Problem 2. Let us consider a time step Δt and denote by $\mathbf{u}_h^n$ the discrete solution at time $t_n = n\Delta t$. Then we denote by N the number of time steps, that is N is the least integer such that $N\Delta t \ge T$.

Problem 3 *Given* $\mathbf{f} \in L^2(]0, L[\times]0, T[)$, *for all* $n = 0, \cdots, N$, *find* $(\mathbf{u}_h^n, p_h^n) \in$

$\mathbf{V}_h \times Q_h$ *and* $\mathbf{X}_h^n \in \mathbf{S}_h$, *such that*

$$< \mathbf{F}_h^{n+1}, \mathbf{v} >_h = \sum_{i=0}^{m} \omega_i \mathbf{f}(s_i, t^{n+1}) \mathbf{v}(\mathbf{X}_{hi}^n), \quad \forall \mathbf{v} \in \mathbf{V}_h \tag{28}$$

$$\begin{aligned} &\rho \left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v} \right) + \mu(\boldsymbol{\nabla}\, \mathbf{u}_h^{n+1}, \boldsymbol{\nabla}\, \mathbf{v}) - (\nabla \cdot \mathbf{v}, p_h^{n+1}) =< \mathbf{F}_h^{n+1}, \mathbf{v} > \forall \mathbf{v} \in \mathbf{V}_h \\ &(\nabla \cdot \mathbf{u}_h^{n+1}, q) = 0 \qquad\qquad \forall q \in Q_h \end{aligned} \tag{29}$$

$$\frac{\mathbf{X}_{hi}^{n+1} - \mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i = 1, \cdots, m \tag{30}$$

$$\mathbf{u}_h^0(\mathbf{x}) = \mathbf{u}_{0h}(\mathbf{x}) \ \forall \mathbf{x} \in \Omega, \tag{31}$$

$$\mathbf{X}_{hi}^0 = \mathbf{X}_0(s_i) \ \forall i = 1, \cdots, m. \tag{32}$$

In Problem 3 we use a modification of Euler scheme. More precisely, first of all we evaluate the right hand side using the known value of $\mathbf{X}_h^n$ at the previous step. Then we compute $\mathbf{u}_h^{n+1}$ by means of the application of the backward Euler scheme to the Stokes equation (23). Then we obtain the new value of $\mathbf{X}_h^{n+1}$ by means of the forward Euler method. The use of the backward Euler scheme to advance in time (23) is not too expensive, since the system is linear and at each step it is possible to use always the same matrices. One could apply backward Euler also to (25), but in this case a non linear system has to be solved at each time step.

We recall that Euler method is only first order. It is also possible to introduce second order methods like Crank-Nicholson for the Stokes equation and a second order Runge-Kutta method or the trapezoidal rule, for a discussion on this topic see [7].

4 Existence of the solution for a one-dimensional problem

In this section we shall prove the existence of the solution of (1)-(4), (5)-(7) in a one-dimensional case via a fixed point argument. Since the divergence free constraint becomes too restrictive in a one-dimensional problem, we consider a standard heat equation instead of the Stokes problem.

Let us set $\Omega =]a, b[$, $f \in H^1(0, T)$, $u_0 \in H_0^1(a, b)$ and $X_0 \in [a, b]$ and consider the following problem:

$$\frac{\partial u}{\partial t} - \mu u_{xx} = F \quad \text{in }]a,b[\times]0,T[\tag{33}$$

$$F(x,t) = f(t)\delta(x - X(t)) \quad \forall x \in]a,b[\,,\ t \in]0,T[\tag{34}$$

$$X'(t) = u(X(t),t) \quad \forall t \in]0,T[\tag{35}$$

$$u(a) = u(b) = 0, \quad u(x) = u_0(x)\ \forall x \in]a,b[\,, \quad X(0) = X_0. \tag{36}$$

Let us introduce the space

$$\mathbb{X} = \left\{ X \in C^0([0,T]) : X(0) = X_0 \right\}. \tag{37}$$

We define an operator $\mathbb{T}$ on $\mathbb{X}$ as follows. Given $\overline{X} \in \mathbb{X}$, let $\overline{F}(x,t) = f(t)\delta(x - \overline{X}(t))$ be the element in $H^{-1}(a,b)$, defined by

$$_{H^{-1}}< \overline{F}(t), v >_{H^1_0} = f(t)v(\overline{X}(t)) \quad \forall v \in H^1_0(a,b)\ \forall t \in]0,T[\,. \tag{38}$$

Then $\overline{u}$ is the solution to the following problem.

$$\text{For } t \in [0,T], \text{ find } u(t) \in H^1_0(a,b) \text{ such that}$$
$$\begin{cases} \dfrac{d}{dt}(u(t),v) + \mu(u_x(t),v_x) = < \overline{F}, v > \forall v \in H^1_0(a,b) \\ u(0) = u_0 \qquad\qquad\qquad\qquad\qquad \text{in }]a,b[\,. \end{cases} \tag{39}$$

Finally, $\overline{\overline{X}} = \mathbb{T}(\overline{X})$ will be the solution X to

$$\begin{cases} X'(t) = \overline{u}(X(t),t) \quad \forall t \in [0,T] \\ X(0) = X_0. \end{cases} \tag{40}$$

We observe that, if $\mathbb{T}$ has a fixed point $\tilde{X}$ in $\mathbb{X}$, then this fixed point and the solution $\tilde{u}$ to (39) corresponding to $\overline{X} = \tilde{X}$, give a solution to (33)-(36).

Standard existence, uniqueness and regularity results for the solution of (39) are summarized in the following lemma, see, e.g. [9].

Lemma 2 *There exists a unique solution $\overline{u}$ to problem (39) such that $\overline{u} \in L^2(0,T;H^1_0(a,b)) \cap C^0([0,T];L^2(a,b))$ and $\overline{u}_t \in L^2(0,T;H^{-1}(a,b))$, with the following energy estimate*

$$\max_{t\in[0,T]} ||\overline{u}(t)||_0^2 + \mu \int_0^T ||\overline{u}(t)||_1^2\, dt \le C \left(||u_0||_0^2 + \int_0^T ||\overline{F}(t)||_{-1}^2 \right). \tag{41}$$

Notice that the norm of $\overline{F}$ in $L^2(0,T;H^{-1}(a,b))$ is bounded by:

$$\left(\int_0^T \|\overline{F}(t)\|_{-1}^2\,dt\right)^{1/2} \leq C\|f\|_{L^2(0,T)}. \tag{42}$$

Moreover, we can prove the following additional regularity result.

Lemma 3 *Let $\overline{X} \in \mathbb{X}$, with $X' \in L^2(0,T)$, and $\overline{u}$ be the solution to (39). Then, there exists a summable function $l(t)$ such that for any points (x,t) and (y,t) in the domain $[a,b]\times[0,T]$*

$$|\overline{u}(x,t)-\overline{u}(y,t)| \leq l(t)|x-y|. \tag{43}$$

Proof. For simplicity, in this proof we shall write $X(t)$ instead of $\overline{X}(t)$.

For any couple of points (x,t) and (y,t) in $]a,b[\times]0,T[$ we can write

$$|\overline{u}(x,t)-\overline{u}(y,t)| = \left|\int_x^y \frac{\partial\overline{u}}{\partial x}(\xi,t)\,d\xi\right| \leq \sup_{\xi\in]a,b[}\left|\frac{\partial\overline{u}}{\partial x}(\xi,t)\right||x-y|. \tag{44}$$

Hence it is enough to show that the supremum on the right hand side is bounded by a summable function $l(t)$.

For all $t\in[0,T]$, let $z(t)$ be the solution of the following second order equation

$$-z_{xx} = \delta(x-X(t)) \;\text{ in }\;]a,b[\,, \quad z(a)=z(b)=0. \tag{45}$$

We can easily compute

$$z(x,t) = \begin{cases} \dfrac{X(t)-b}{a-b}(x-a) & \text{for } a\leq x\leq X(t) \\ \dfrac{X(t)-a}{a-b}(x-b) & \text{for } X(t)\leq x\leq b \end{cases} \tag{46}$$

We observe that $z(t)$ is a Lipschitz continuous function for all $t\in[0,T]$. Then we can write the solution of (39) as the sum $\overline{u}(x,t) = w(x,t) + f(t)z(x,t)$, where w is the solution of the following problem:

$$\begin{aligned} & w_t - w_{xx} = -f'z + fz_t \qquad \text{in }]a,b[\times]0,T[\,, \\ & w(a,t) = w(b,t) = 0, \qquad 0\leq t\leq T, \\ & w(x,0) = u_0(x) - f(0)z(0) \qquad a\leq x\leq b. \end{aligned} \tag{47}$$

The function in the right hand side belongs to $L^2(]a,b[\times]0,T[)$ with

$$\|-f'z+fz_t\|_0 \leq C\left(\|f'\|_{L^2(0,T)} + \sup_{t\in]0,T[}|f(t)|\|X'\|_{L^2(0,T)}\right).$$

Due to standard regularity results for parabolic equations like (47), we obtain that $w \in L^2(0,T;H^2(a,b))$ with the bound

$$\|w\|_{L^2(0,T;H^2(a,b))} \leq C\left(\|f'\|_{L^2(0,T)} + \sup_{t\in]0,T[} |f(t)| \|X'\|_{L^2(0,T)}\right).$$

Thanks to Sobolev embedding theorem we have that $w \in L^2(0,T;W^{1,\infty}(a,b))$ and that also $\overline{u} \in L^2(0,T;W^{1,\infty}(a,b))$. Then the term $\sup_{\xi\in]a,b[}\left|\frac{\partial \overline{u}}{\partial x}(\xi,t)\right|$ in (44) belongs to $L^2(0,T)$ and (43) is proved. □

The properties of $\overline{u}$ stated in the previous two lemmas imply that problem (40) is a standard well-posed Carathéodory differential equation and we can state the existence and uniqueness of its solution (see, e.g. [2]).

Theorem 1 *Let $\overline{u}$ be the solution of (39), then we have:*

i) for all $t_0 \in [0,T]$ and $y_0 \in]a,b[$ there exists a closed interval $[t_0, t_0+d]$ with $d > 0$ such that the problem

$$\dot{y} = \overline{u}(y,t), \quad y(t_0) = y_0 \tag{48}$$

has a solution;

ii) in the domain $[a,b] \times [0,T]$ there exists at most one solution of problem (48) in the sense that, if there exist two solutions, whose graphs lie in $[a,b] \times [0,T]$, these solutions coincide on the common part of their intervals of existence;

iii) for all $t_0 \in [0,T]$ and any $y_0 \in [a,b]$ there exists a unique solution $y(t)$ to (48) defined in $[0,T]$.

Proof. In the domain $[a,b] \times [0,T]$, the Carathéodory conditions which imply $i)$ are:

a) the function $\overline{u}(y,t)$ is defined and continuous in y for almost all t;
b) the function $\overline{u}(y,t)$ is measurable in t for each y;
c) $|\overline{u}(y,t)| \leq m(t)$, where $m(t)$ belongs to $L^1(0,T)$.

The required properties are consequences of the regularity results for problem (39) reported in Lemma 2. In fact $\overline{u} \in L^2(0,T;H_0^1(a,b))$ and this implies that for almost every $t \in]0,T[$ $\overline{u}$ is continuous with respect to y and that it is measurable in t for each y. Moreover, we have that

$$|\overline{u}(y,t)| \leq \sup_{y\in[a,b]} |\overline{u}(y,t)| \leq C\|\overline{u}_x(t)\|_1$$

which gives c).

The Lipschitz continuous property stated for the solution $\overline{u}$ to problem (39) in Lemma 3 is a sufficient condition for the uniqueness of the solution of (48), that is *ii*).

In order to prove *iii*), we observe that, due to the boundary condition enforced in problem (39), $\overline{u}(t)$ vanishes for $x = a$ and $x = b$ for all t; a solution y to (48) is constant either if $y_0 = a$ or if $y_0 = b$. Hence, if $y_0 \in \,]a, b[$, it can never touch the boundary $\partial\Omega$ (otherwise the uniqueness would fail). $\square$

Theorem 1 gives that the solution $\overline{\overline{X}}$ to (40) enjoins the properties stated in the following corollary.

Corollary 1 *Let $X_0 \in \,]a, b[$. There exists a unique solution $\overline{\overline{X}}$ to equation (40) defined in $[0, T]$, with $\dfrac{d\overline{\overline{X}}}{dt} \in L^2(0, T)$ and it holds*

$$\overline{\overline{X}}(t) \in \,]a, b[\qquad \forall t \in [0, T], \qquad \left\| \frac{d\overline{\overline{X}}}{dt} \right\|_{L^2(0,T)} \leq K. \tag{49}$$

Proof. The first property in (49) is a consequence of the uniqueness result stated in *iii*) of Theorem 1. We prove the second one. Since $\overline{u}(a, t) = \overline{u}(b, t) = 0$ we have that $\overline{u}(\overline{\overline{X}}(t), t) = \overline{u}(\overline{\overline{X}}(t), t) - u(a, t)$. Hence we can compute

$$\left|\overline{u}(\overline{\overline{X}}(t), t)\right| = \int_a^{\overline{\overline{X}}(t)} \overline{u}_x(\xi, t)\, d\xi \leq C(\overline{\overline{X}}(t) - a)^{1/2} \|\overline{u}_x(\cdot, t)\|_{L^2(a,b)}. \tag{50}$$

Hence we have

$$\left\| \frac{d\overline{\overline{X}}}{dt} \right\|^2_{L^2(0,T)} = \int_0^T \left|\overline{u}(\overline{\overline{X}}(t), t)\right|^2 \, dt \leq K$$

where K depends on the length of the interval $]a, b[$, on $\|u_0\|_0$ and on $\|f\|_{L^2(0,T)}$, see (41). $\square$

We have now all the elements to prove the main result of this section that is the existence of a fixed point of the operator $\mathbb{T}$ which in turn is a solution of problem (33)-(36).

Theorem 2 *Given K as in (49), let us introduce the subset of $\mathbb{X}$ defined as follows:*

$$B = \{Y \in \mathbb{X} : Y(t) \in [a, b], \ \|Y'\|_{L^2(0,T)} \leq K\}. \tag{51}$$

Then there exists a fixed point of $\mathbb{T}$ in B.

Proof. The set B is a convex and compact subset of $\mathbb{X}$, which is a Banach space. The operator $\mathbb{T}$ maps B into itself, thanks to (49). It remains to show that $\mathbb{T}$ is continuous, that is

$$\begin{aligned}&\forall\varepsilon>0, \text{ there exists } \chi>0 \text{ such that}\\ &\|\mathbb{T}X-\mathbb{T}Y\|_{C^0(0,T)}\le\varepsilon \quad \text{for all } X,Y\in B \text{ with } \|X-Y\|_{C^0(0,T)}\le\chi.\end{aligned}$$

Given two elements X and Y in B, we denote by u_X and u_Y the corresponding solution of (39). We have that the difference u_X-u_Y solves the following equation

$$\begin{aligned}&\frac{d}{dt}((u_X-u_Y)(t),v)+\mu((u_X-u_Y)_x(t),v_x)=f(t)(v(X(t))\ -v(Y(t)))\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\forall v\in H_0^1(a,b)\\ &(u_X-u_Y)(0)=0 \qquad\qquad\qquad\qquad\qquad\qquad\quad \text{in }]a,b[\,.\end{aligned}$$

We take in the above equation $v=u_X-u_Y$ and we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\ \|(u_X-u_Y)(t)\|_0^2+\mu\|(u_X-u_Y)_x(t)\|_0^2\\ &\quad=f(t)\left((u_X-u_Y)(X(t),t)-(u_X-u_Y)(Y(t),t)\right)\\ &\quad=f(t)\int_{Y(t)}^{X(t)}(u_X-u_Y)_x(\xi,t)\,d\xi\\ &\quad\le C|f(t)|\,|X(t)-Y(t)|^{1/2}\|(u_X-u_Y)_x(t)\|_0\end{aligned}$$

Integrating from 0 to T we obtain

$$\sup_{t\in]0,T[}\|(u_X-u_Y)(t)\|_0^2+\mu\int_0^T\|(u_X-u_Y)_x(t)\|_0^2\le C\sup_{t\in]0,T[}|X(t)-Y(t)|. \tag{52}$$

Let us denote by $\tilde{X}=\mathbb{T}X$ and $\tilde{Y}=\mathbb{T}Y$ the solution of (40) with right hand side u_X and u_Y respectively. Hence we have

$$\begin{aligned}\tilde{X}'(t)-\tilde{Y}'(t)&=u_X(\tilde{X}(t),t)-u_Y(\tilde{Y}(t),t))\\ &=u_X(\tilde{X}(t),t)-u_Y(\tilde{X}(t),t)+u_Y(\tilde{X}(t),t)-u_Y(\tilde{Y}(t),t)\end{aligned}$$

Integrating form 0 to t and using the fact that u_Y is Lipschitz continuous (see (43)) we have

$$\tilde{X}(t)-\tilde{Y}(t)=\int_0^t(u_X-u_Y)(\tilde{X}(\tau),\tau)\,d\tau+\int_0^t\Big(u_Y(\tilde{X}(\tau),\tau)-u_Y(\tilde{Y}(\tau),\tau)\Big)\,d\tau.$$

Using (50) and (43) gives

$$|\tilde{X}(t)-\tilde{Y}(t)|\le Ct^{1/2}\left(\int_0^T\|(u_X-u_Y)_x(t)\|_0^2\,dt\right)^{1/2}+C\int_0^t l(t)|\tilde{X}(\tau)-\tilde{Y}(\tau)|\,d\tau.$$

Hence the Gronwall's lemma and (52) imply

$$\sup_{t\in]0,T[} |(\tilde{X}-\tilde{Y})(t)| \leq C \sup_{t\in]0,T[} |(X-Y)(t)|^{1/2},$$

and this gives the required continuity of $\mathbb{T}$.

Hence $\mathbb{T}$ satisfies the assumptions of the Schauder theorem and admits at least a fixed point. $\square$

5 Numerical results for the one-dimensional problem

In this section we will show the results of some computations which could give us some indication for further investigations.

Let us divide the interval $[a,b]$ in N_x equal parts of length $h=(b-a)/N_x$ and denote by $x_i = a+ih$, for $i=0,\cdots,N_x$ the subdivision points. We consider the following finite dimensional subspace V_h of $H_0^1(a,b)$

$$V_h = \{v \in H_0^1(a,b) : v \text{ piecewise linear}\}.$$

Next, let us choose a time step Δt and let $N_t = T/\Delta t$. We use the following notation $u_h^n = u_h(n\Delta t)$. Then we shall apply the following numerical scheme to solve (33)-(36).

1. Let $u_h^0 \in V_h$ be the interpolation of u_0 in V_h, hence u_h^0 is the piecewise linear function on $[a,b]$ such that $u_h^0(x_i) = u_0(x_i)$ for $i=0,\cdots,N_x$.
2. Let $X_h^0 = X_0$.
3. For $n=0,\cdots,N_t-1$, compute u_h^{n+1} and X_h^{n+1} as follows:
   3.1 find $u_h^{n+1} \in V_h$ such that

$$(u_h^{n+1}, v_h) + \Delta t \mu (u_{hx}^{n+1}, v_{hx}) = (u_h^n, v_h) + \Delta t\, f((n+1)\Delta t) v_h(X_h^n) \quad \forall v_h \in V_h$$

   3.2 find X_h^{n+1} such that

$$X_h^{n+1} = X_h^n + \Delta t\, u_h^{n+1}(X_h^n).$$

With the scheme above we have computed the solution with $[a,b]=[-1,1]$, $T=10$, $\mu=1$, $X_0=0$ and two different functions f:

$$f(t) = \frac{\sin t}{2} \qquad 0 \leq t \leq 10 \tag{53}$$

$$f(t) = 1 \qquad 0 \leq t \leq 10. \tag{54}$$

In the first test we have $h = 1/50$ and $\Delta t = 1/10$. For f given, respectively, by (53) and (54), Fig.s 1 and 3 display the behavior of u_h^n for several values of t_n from $t = .1$ to $t = 9.9$, while Fig.s 2 and 4 contain the computed X_h on the interval $[0, 10]$.

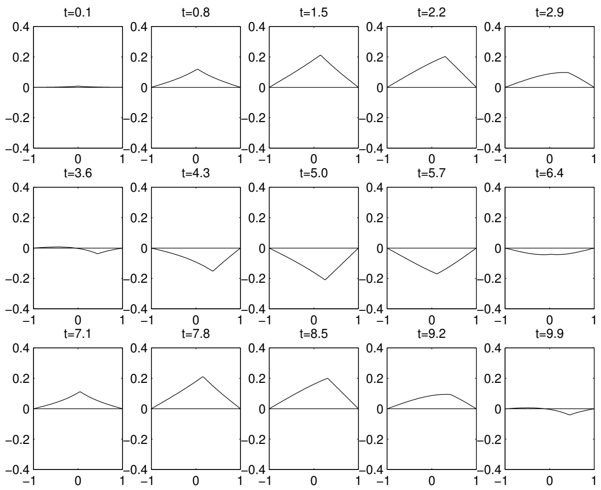

Fig. 1. The evolution of u_h with $f = \sin t/2$.

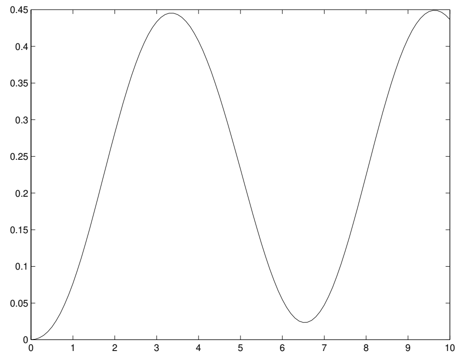

Fig. 2. The graph of X_h with $f = \sin t/2$.

We can observe that in the case of a periodic f like (53) we have an almost periodic behavior of the solution. While in the case of f constant and positive, since u is always positive, we have that X_h is increasing; moreover it tends asymptotically to 1 and u tends to the function identically 0.

The next pictures contain a comparison of two solutions corresponding to different initial values for X_h, $X_0 = 0$ and $X_0 = \Delta X_0$. More precisely, for the two choices of f, (53) and (54), respectively, Fig.s 5 and 7 contain the comparison of the computed u_h, while Fig.s 6 and 8 display the graphs of X_h corresponding to the different initial values. In Fig.s 6 and 8 we notice that the solution which is greater at the initial instant remains greater in the whole time interval. Moreover we can see that the difference between the two solution does not increase in time. Table 1 and 2 display the norms of the difference on u_h and on X_h corresponding to various perturbations ΔX_0 of the initial datum

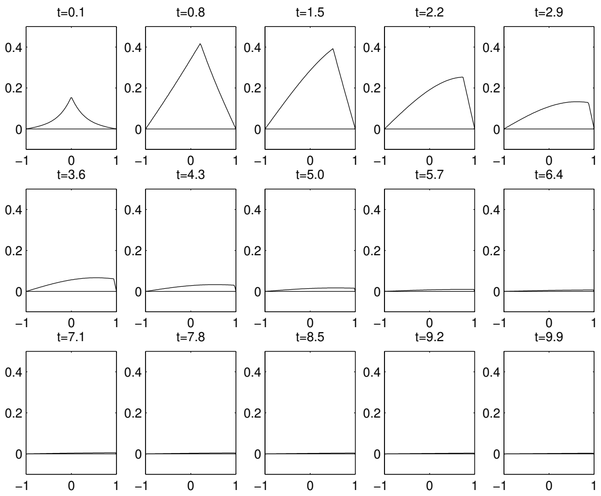

Fig. 3. The evolution of u_h with $f = 1$.

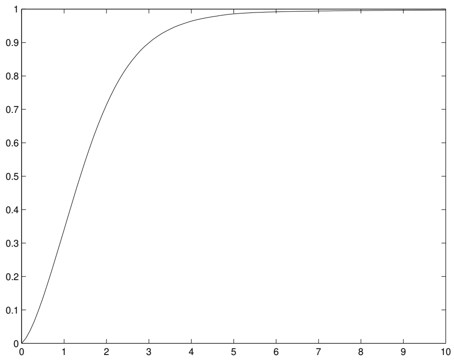

Fig. 4. The graph of X_h with $f = 1$.

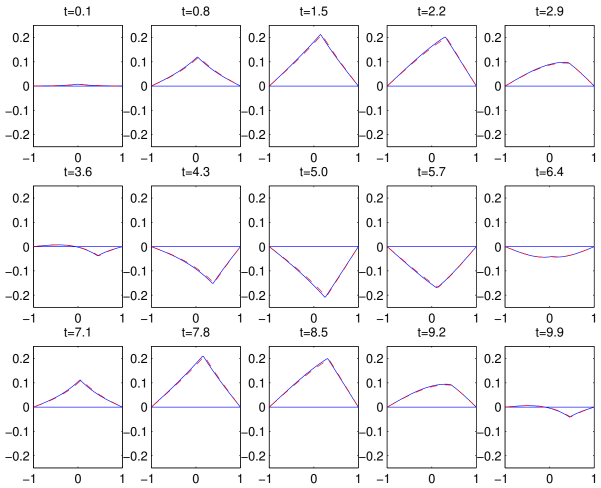

Fig. 5. The comparison of u_h with $f = \sin t/2$ corresponding to two different initial values X_0. The dashed lines correspond to the perturbed datum.

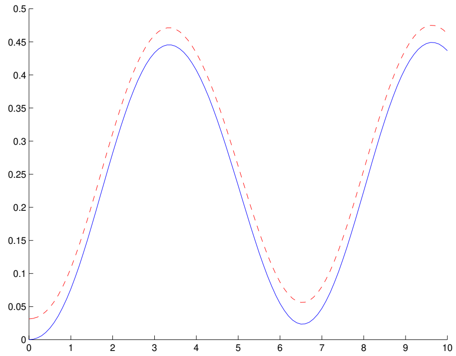

Fig. 6. The comparison of X_h with $f = \sin t/2$ corresponding to two different initial values X_0. The dashed line corresponds to the perturbed datum.

Table 1
Errors in u_h and X_h corresponding to different perturbations on the initial datum X_0 for different values of the mesh size h and the time step Δt. $f = \sin t/2$

	$h = 0.04$ $\Delta t = 0.04$		$h = 0.02$ $\Delta t = 0.02$		$h = 0.01$ $\Delta t = 0.01$	
ΔX_0	err. u_h	err. X_h	err. u_h	err. X_h	err. u_h	err. X_h
0.1571	0.0293779	0.1573866	0.0294095	0.1570798	0.0294092	0.1570816
0.0628	0.0121962	0.0629751	0.0123005	0.0632567	0.0122187	0.0628618
0.0314	0.0063138	0.0316661	0.0062655	0.0317612	0.0061965	0.0314334
0.0157	0.0031717	0.0160608	0.0031855	0.0159313	0.0031411	0.0157971

X_0 for different values of the mesh size h and the time step Δt. One can see that the perturbations on the solutions are proportional to the perturbations on the initial value. In particular, notice that the perturbation on X_h is not amplified during the evolution.

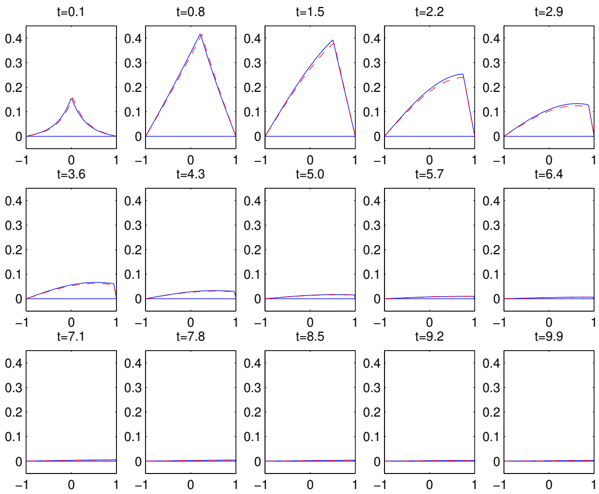

Fig. 7. The comparison of u_h with $f = 1$ corresponding to two different initial values X_0. The dashed lines correspond to the perturbed datum.

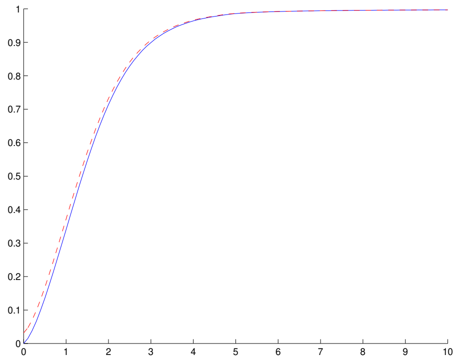

Fig. 8. The comparison of X_h with $f = 1$ corresponding to two different initial values X_0. The dashed line corresponds to the perturbed datum.

Table 2
Errors in u_h and X_h corresponding to different perturbations on the initial datum X_0 for different values of the mesh size h and the time step Δt. $f = 1$.

	$h = 0.04$ $\Delta t = 0.04$		$h = 0.02$ $\Delta t = 0.02$		$h = 0.01$ $\Delta t = 0.01$	
ΔX_0	err. u_h	err. X_h	err. u_h	err. X_h	err. u_h	err. X_h
0.1571	0.0598120	0.1570791	0.0598606	0.1570813	0.0599150	0.1570864
0.0628	0.0244720	0.0631801	0.0244200	0.0629495	0.0244391	0.0628909
0.0314	0.0125532	0.0314769	0.0124230	0.0315347	0.0124385	0.0314497
0.0157	0.0065090	0.0163979	0.0062861	0.0157297	0.0062777	0.0157453

References

[1] F. Brezzi and M. Fortin, *Mixed and hybrid finite element methods*, Springer–Verlag, New York, 1991.

[2] A. F. Filippov, *Differential equations with discontinuous right hand sides*, Kluwer Academic Publishers Group, Dordrecht, 1988, Translated from the Russian.

[3] E. Givelberg, *Modeling elastic shells immersed in fluid*, Ph.D. thesis, New York University, 1997.

[4] M.-C. Lai and C. S. Peskin, *An immersed boundary method with formal second-order accuracy and reduced numerical viscosity*, J. Comput. Phys. **160** (2000), no. 2, 705–719.

[5] D.M. Mc Queen and C.S. Peskin, *Heart simulation by an immersed boundary method with a formal second-order accuracy and reduced numerical viscosity*, Mechanics for a New Millennium, Proceedings of the International Conference on Theoretical and Applied Mechanics (ICTAM) 2000 (H. Aref and J.W. Phillips, eds.), Kluwer Academic Publisher, 2001.

[6] D.M. Mc Queen, C.S. Peskin, and L. Zhu, *The immersed boundary method for incompressible fluid-structure interaction*, Proceedings of the First M.I.T. Conference on Computational Fluid and Solid Mechanics, June 12-15, 2001 (Oxford, UK) (K. J. Bathe, ed.), Vol. 1, Elsevier Science Ltd., 2001, pp. 26–30.

[7] C. S. Peskin, *The immersed boundary method*, Acta Numerica, 2002, Cambridge University Press, 2002.

[8] C. S. Peskin and B. F. Printz, *Improved volume conservation in the computation of flows with immersed elastic boundaries*, J. Comput. Phys. **105** (1993), no. 1, 33–46.

[9] A. Quarteroni and A. Valli, *Numerical approximation of partial differential equations*, Springer-Verlag, Berlin, 1994.

[10] M. E. Rosar and C. S. Peskin, *Fluid flow in collapsible elastic tubes: a three-dimensional numerical model*, New York J. Math. **7** (2001), 281–302 (electronic).