Finite element method for advection diffusion

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Outline

1. Computational fluid dynamics
2. Advection diffusion equation

Navier-Stokes equations

The equations governing the dynamics of incompressible fluids are:

Navier-Stokes equation

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+(\nabla \mathbf{u})\mathbf{u}\right)-\operatorname{div}\mu\mathcal{E}(u)+\nabla p=\rho\mathbf{b}\ in\Omega$$

$$\operatorname{div}\mathbf{u}=0.$$

where $\mathcal{E}(u)=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^{\top})$

Cauchy stress tensor: $\sigma=-\nabla p\mathbb{I}+\mu\mathcal{E}(u)$.

Advection diffusion equation

Equazione differenziale completa

$$
\begin{aligned}
&-\operatorname{div}(K\nabla u) + \beta \cdot \nabla u + \sigma u = f && \text{in } \Omega \\
&u = 0 && \text{on } \partial\Omega
\end{aligned}
$$

Weak problem

Find $u \in H_0^1(a,b)$ such that

$$a(u,v) = F(v) \quad \forall v \in H_0^1(\Omega),$$

with

$$a(u,v) = \int_\Omega (K\nabla u \nabla v + \beta \nabla u v + \sigma u v)\, dx$$

$$F(v) = \int_\Omega f v\, dx \quad \forall v \in H_0^1(\Omega)$$

Finite element discretization

Given a regular triangulation $\mathcal{T}_h$, let np be the number of vertexes, ne number of the elements. We set

$$X_h^r = \{v \in C^0(\overline{\Omega}) : v|_K \in \mathbb{P}_r(K),\ \forall K \in \mathcal{T}_h\}$$
$$V_h = \{v \in X_h^r : v = 0 \text{ on } \Omega\}.$$

Discrete problem

Find $u_h \in V_h$ such that

$$a(u_h, v_h) = F(v_h) \quad \forall v_h \in V_h.$$

Let φ_i for $i = 1, \ldots, N(h)$ be the basis functions of V_h. Then

$$u_h(x) = \sum_{i=1}^{N(h)} u_j \varphi_j(x).$$

Taking $v_h = \varphi_i$ in the discrete equation we obtain

$$\sum_{j=1}^{N(h)} u_j \, a(\varphi_j, \varphi_i) = F(\varphi_i) \quad i = 1, \ldots, N(h).$$

Hence we have to solve the linear system $A\underline{\mathbf{u}} = \underline{\mathbf{F}}$ with

$$A_{ij} = \int_\Omega (K \nabla\varphi_j \nabla\varphi_i + \beta\nabla\varphi_j\varphi_i + \sigma\varphi_j\varphi_i) \, dx$$

The matrix A is not symmetric anymore.

Construction of the matrix

We use the following array provided by Matlab: `p,t,e`.

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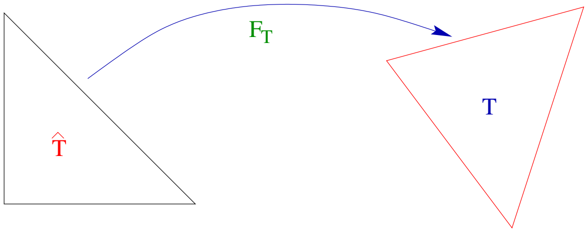
The function `shape2D` contains the analytical expression of the basis functions on the reference element

`shape2D(i,k,x,y)`

$i = 1, 2, 3,$
$k = 0$ for the function,
$k = 1$ partial derivative with respect to x
$k = 2$ partial derivative with respect to y



$$F_T(\underline{\hat{x}}) = \underline{a} + B_T\underline{\hat{x}}$$

We construct the matrix and the right hand side element by element. For each element we compute a local matrix 3×3 and a local vector by transforming the integrals into integrals on the reference element.
The change of variable in the derivatives gives:

$$\frac{\partial \varphi_i}{\partial x} = \frac{\partial \hat{\varphi}_i}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\varphi}_i}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial x}$$
$$\frac{\partial \varphi_i}{\partial y} = \frac{\partial \hat{\varphi}_i}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\varphi}_i}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial y}$$

and

$$\frac{\partial \hat{x}}{\partial x}, \ \frac{\partial \hat{y}}{\partial x}, \ \frac{\partial \hat{x}}{\partial y}, \ \frac{\partial \hat{y}}{\partial y}$$

are given by the element of the inverse of the Jacobian B_T.

Hence it holds:

$$\int_T \nabla\varphi_j \nabla\varphi_i dx = \int_{\hat{T}} \nabla_{\hat{x}}\hat{\varphi}_i^\top B_T^{-1} B_T^{-\top} \nabla_{\hat{x}}\hat{\varphi}_j J d\hat{x}$$

$$\int_T \beta\cdot\nabla\varphi_j\varphi_i dx = \int_{\hat{T}} \beta\cdot B_T^{-\top}\nabla_{\hat{x}}\hat{\varphi}_j\hat{\varphi}_i J d\hat{x}$$

$$\int_T \varphi_j\varphi_i dx = \int_{\hat{T}} \hat{\varphi}_j\hat{\varphi}_i J d\hat{x}$$

$$\int_T f\varphi_i dx = \int_{\hat{T}} f(F_T(\underline{\hat{x}}))\hat{\varphi}_i J d\hat{x}$$