Boundary value problems

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Outline

Poisson equation

We consider the differential equation

$$\begin{aligned} &-\mu u''(x) = f(x) \quad \text{per } x \in (a, b) \\ &u(a) = \alpha, \quad u(b) = \beta, \end{aligned}$$

with $\mu > 0$, and f continuous function on (a, b).

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Theorem

There exists a unique solution of the Poisson equation.

Finite difference discretization

Let us subdivide the interval $[a, b]$ into $N + 1$ equal parts, set $h = (b - a)/(N + 1)$, and $x_i = a + ih$ for $i = 0, \ldots, N + 1$.

The second derivative can be discretized by means of the finite difference of second order

$$u''(x_i) \approx \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2} \quad \text{for } i = 1, \ldots, N.$$

Denoting by u_i^h the approximate value of $u(x_i)$, we obtain the following linear system:

$$-\mu \frac{u_{i+1}^h - 2u_i^h + u_{i-1}^h}{h^2} + \sigma(x_i)u_i^h = f(x_i) \quad \text{for } i = 1, \ldots, N$$
$$u_0^h = \alpha \qquad u_{N+1}^h = \beta$$

The discrete solution u_i^h for $i = 1, \ldots, N$ is obtained by solving the linear system

$$Au^h = F$$

where A is the tridiagonal matrix with the following entries

$$a_{ii} = 2\mu/h^2 + \sigma(x_i), \quad a_{ii-1} = a_{ii+1} = -\mu/h^2$$

and the right hand side is given by

$$\begin{aligned} F_1 &= f(x_1) + \mu\alpha/h^2, \\ F_i &= f(x_i) \text{ for } i = 2, \ldots, N-1, \\ F_N &= f(x_N) + \mu\beta/h^2. \end{aligned}$$

Error estimate

$$\max_{1 \le i \le N} |u(x_i) - u_i^h| \le Ch^2 \max_{a \le x \le b} |u^{(4)}(x)|$$

Construction of the matrix

The matrix A has the following form:

$$K = \frac{\mu}{h^2} \begin{bmatrix} 2 & -1 & \dots & \dots & \dots & \dots & 0 \\ -1 & 2 & -1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 2 & -1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -1 & 2 & -1 \\ 0 & \dots & \dots & \dots & \dots & -1 & 2 \end{bmatrix}$$

Construction of the right hand side

The right hand side is the sum $F = \tilde{F} + bc$ where $\tilde{F}$ takes into account f, while bc is relative to the boundary conditions

$$\tilde{F} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ \dots \\ \dots \\ f(x_N) \end{bmatrix} \qquad bc = \begin{bmatrix} \mu\alpha/h^2 \\ 0 \\ \dots \\ 0 \\ \mu\beta/h^2 \end{bmatrix}$$

Computationally, it is not convenient to construct the vector bc, but the result can be achieved by summing to the first and last components of F the contribution deriving from the boundary conditions.

Function for the resolution of the Poisson equation

The function eqlim computes the solution of the Poisson equation

$$-\mu u''(x) = f(x) \quad \text{per } x \in [a, b]$$
$$u(a) = \alpha, \quad u(b) = \beta.$$

by means of the following instruction

```
[x,u]=eqlim(f,mu,a,b,alfa,beta,N)
```

Input

`f`	name of the function_handle containing the analytic expression of f
`mu`	coefficient in front of the second derivative
`a,b`	endpoints of the interval
`alfa,beta`	values at the endpoints
`N`	number of the grid points

Output

`x`	grid points
`u`	solution

Function `eqlim`

The function `eqlim` consists of 5 steps:

- assignment of the grid: computation of h and of the grid points $x_i = a + ih$ for $i = 1, \ldots, N$;
- construction of A by using the functions `ones` e `diag`;
- construction of the right hand side;
- solution of the linear system;
- organization of the output.

Oss The vectors `x` e `u` are relative to the internal grid points, in order to repreent the solution it is useful to add the values at the endpoints. To this aim it is possible to add components to the vectors as follows

- `x=[a,x,b]` if `x` is a row vector;
- `u=[alfa;u;beta]` if `u` is a column vector.

Exercise 1

Solve the differential equation

$$\begin{aligned} &-u''(x) = -2\cos x e^x \quad \text{for } x \in [0,\pi] \\ &u(0) = u(\pi) = 0. \end{aligned}$$

The exact solution is $u(x) = \sin x \, e^x$.

Write a script file which:

- assigns the data;
- computes the solution by using the function `eqlim`;
- plots the discrete solution together with the continuous one;
- computes the error

$$E = \max_{1\leq i\leq N} |u(x_i) - u_i^h|.$$

Error The error can be computed using the command `norm`.

Convergence of the finite difference scheme

Exercise 2

Given `N=[10 20 40 80 160 320]`, for each value of N

- compute the solution of the differential equation in Exercise 1;
- plot the discrete solution together with the continuous one;
- compute the error.

At the end plot the errors with respect to h using bilogarithmic scale.
Compute the rate of convergence.

Exercise 3

Given $q \geq 2$, consider the differential equation

$$-u''(x) = q(q-1)|x|^{q-2} \text{ for } x \in (-1,1), \qquad u(-1) = u(1) = 0.$$

The exact solution is $u(x) = 1 - |x|^q$.
Compute the rate of convergence of the finite difference scheme for the following values of q:

$$q = 2, 2.5, 3, 3.5, 4.$$

If $q < 2$, how does the solution behave?

Esercizio 4

Consider the following differential equation

$$-u''(x) = 12x^2 \text{ per } x \in (-1, 1) \quad u(-1) = u(1) = 0.$$

The exact solution is $u(x) = 1 - x^4$.
Compute the solution corresponding to the following values of `N`

```
N=[10 20 40 80 160 320 640 1280];
N=[N 2500 5000 10000 20000 40000 80000 1.e5 1.2e5
1.5e5];
```

Compute the error as a function of h and plot it with bilogarithmic scale.