Finite element method for advection diffusion

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1. Computational fluid dynamics
2. Advection diffusion equation
The equations governing the dynamics of incompressible fluids are:

**Navier-Stokes equation**

\[
\rho \left( \frac{\partial \mathbf{u}}{\partial t} + (\nabla \mathbf{u}) \mathbf{u} \right) - \text{div} \mu \mathcal{E}(u) + \nabla p = \rho \mathbf{b} \text{ in } \Omega
\]
\[
\text{div } \mathbf{u} = 0.
\]

where \( \mathcal{E}(u) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^\top) \)

**Cauchy stress tensor:** \( \sigma = -\nabla p \mathbb{I} + \mu \mathcal{E}(u). \)
Advection diffusion equation

Equazione differenziale completa

\[- \text{div}(K \nabla u) + \beta \cdot \nabla u + \sigma u = f \quad \text{in } \Omega\]
\[u = 0 \quad \text{on } \partial \Omega\]

Weak problem

Find \( u \in H^1_0(a, b) \) such that

\[ a(u, v) = F(v) \quad \forall v \in H^1_0(\Omega), \]

with

\[ a(u, v) = \int_\Omega (K \nabla u \nabla v + \beta \nabla u v + \sigma u v) \, dx \]
\[ F(v) = \int_\Omega f v \, dx \quad \forall v \in H^1_0(\Omega) \]
Given a regular triangulation $\mathcal{T}_h$, let $np$ be the number of vertexes, $ne$ number of the elements. We set

$$X_h^r = \{ v \in C^0(\overline{\Omega}) : v|_K \in \mathbb{P}_r(K), \forall K \in \mathcal{T}_h \}$$

$$V_h = \{ v \in X_h^r : v = 0 \text{ on } \Omega \}.$$
Let $\varphi_i$ for $i = 1, \ldots, N(h)$ be the basis functions of $V_h$. Then

$$u_h(x) = \sum_{i=1}^{N(h)} u_j \varphi_j(x).$$

Taking $v_h = \varphi_i$ in the discrete equation we obtain

$$\sum_{j=1}^{N(h)} u_j a(\varphi_j, \varphi_i) = F(\varphi_i) \quad i = 1, \ldots, N(h).$$

Hence we have to solve the linear system $A u = F$ with

$$A_{ij} = \int_{\Omega} (K \nabla \varphi_j \nabla \varphi_i + \beta \nabla \varphi_j \varphi_i + \sigma \varphi_j \varphi_i) \, dx$$

The matrix $A$ is not symmetric anymore.
Construction of the matrix

We use the following array provided by Matlab: \( p, t, e \).
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The function \texttt{shape2D} contains the analytical expression of the basis functions on the reference element \( \texttt{shape2D}(i,k,x,y) \)

\( i = 1, 2, 3, \)
\( k = 0 \) for the function,
\( k = 1 \) partial derivative with respect to \( x \)
\( k = 2 \) partial derivative with respect to \( y \)

\[
F_T(\hat{x}) = a + B_T \hat{x}
\]
We construct the matrix and the right hand side element by element. For each element we compute a local matrix $3 \times 3$ and a local vector by transforming the integrals into integrals on the reference element.

The change of variable in the derivatives gives:

\[
\frac{\partial \varphi_i}{\partial x} = \frac{\partial \hat{\varphi}_i}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} + \frac{\partial \hat{\varphi}_i}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial x}
\]

\[
\frac{\partial \varphi_i}{\partial y} = \frac{\partial \hat{\varphi}_i}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial y} + \frac{\partial \hat{\varphi}_i}{\partial \hat{y}} \frac{\partial \hat{y}}{\partial y}
\]

and

\[
\frac{\partial \hat{x}}{\partial x'}, \frac{\partial \hat{y}}{\partial x'}, \frac{\partial \hat{x}}{\partial y'}, \frac{\partial \hat{y}}{\partial y'}
\]

are given by the element of the inverse of the Jacobian $B_T$. 
Hence it holds:

\[ \int_T \nabla \varphi_j \nabla \varphi_i \, dx = \int_{\hat{T}} \nabla_{\hat{x}} \hat{\varphi}_i^T B_T^{-1} B_T^{-T} \nabla_{\hat{x}} \hat{\varphi}_j \, J d\hat{x} \]

\[ \int_T \beta \cdot \nabla \varphi_j \varphi_i \, dx = \int_{\hat{T}} \beta \cdot B_T^{-T} \nabla_{\hat{x}} \hat{\varphi}_j \hat{\varphi}_i \, J d\hat{x} \]

\[ \int_T \varphi_j \varphi_i \, dx = \int_{\hat{T}} \hat{\varphi}_j \hat{\varphi}_i \, J d\hat{x} \]

\[ \int_T f \varphi_i \, dx = \int_{\hat{T}} f(F_T(\hat{x})) \hat{\varphi}_i \, J d\hat{x} \]