Finite Element Approximation of Eigenvalue Problems
Lesson 4

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Outline

Some references

Babuška- Rheinboldt 1978 introduction of an a posteriori error estimator 1D
Verfürth 1994 upper bound of suboptimal order in the approximation of the eigenvalue and only a global lower bound
Alonso-Dello Russo-Vampa 1999,2000 error estimator of residual type
Larson 2000 optimal order estimates assuming the H^2-regularity of the eigenfunctions
Heuveline and Rannacher 2001 obtained results based on a general analysis for nonlinear equations without requiring the H^2 -regularity of the problem
Mao-Shen-Zhou 2006 Adaptive finite element algorithms for eigenvalue problems based on local averaging type a posteriori error estimates
Durán-Padra-Rodríguez 2003 simple analysis for a residual type error estimator for the linear finite element approximation of a second order elliptic problem

Some references mixed eigenproblem

Durán-G.-Padra 1999 analyzed an error estimator for the approximation of the eigensolutions of a second- order elliptic problem by using the equivalence between the mixed finite element method of Raviart-Thomas of the lowest order and the non-conforming piecewise linear approximation of Crouzeix and Raviart

Alonso-Dello Russo-Otero-Souto-Padra-Rodríguez 2000, Alonso-Dello Russo-Padra-Rodríguez 2001 error estimator to deal with structural-acoustic vibration problems on matching and non matching grids, respectively

Gardini 2004, 2005 A posteriori error estimators for eigenvalue problems in mixed form studied without using the equivalence with a non-conforming approximation

Lovadina-Lyly-Stenberg 2009 Stokes eigenvalue problem

Convergence and optimality of the adaptive method

Dai-Xu-Zhou 2008 Convergence and optimal complexity of adaptive finite element eigenvalue computations
Giani-Graham 2009 A convergent adaptive method for elliptic eigenvalue problems under the condition that the initial mesh is sufficiently fine
Garau-Morin 2009 Convergence and quasi-optimality of adaptive FEM for Steklov eigenvalue problems
Garau- Morin-Zuppa 2009 Convergence of adaptive finite element methods for eigenvalue problems
Gallistl 2014 An optimal adaptive FEM for eigenvalue clusters
Boffi-Gallistl-Gardini-G 2017 Optimal convergence of adaptive FEM for eigenvalue clusters in mixed form
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Outline

References

A posteriori estimates for eigenvalue problem

AFEM for eigenvalue problems in mixed form

Numerical results

Laplace eigenvalue problem

$V = H_0^1(\Omega)$, $a(u,v) = (\nabla u, \nabla v)$

Continuous problem

Find $\lambda \in \mathbb{R}$ and $u \in V$ with $u \neq 0$ such that

$$\begin{cases} a(u,v) = \lambda(u,v) & \forall v \in V \\ \|u\|_0 = 1 \end{cases}$$

Discrete problem

Find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$ with $u_h \neq 0$ such that

$$\begin{cases} a(u_h,v) = \lambda_h(u_h,v) & \forall v \in V_h \\ \|u_h\|_0 = 1 \end{cases}$$

Orthogonality

Error equation for the source problem

$$a(u - u_h, v) = 0 \quad \forall v \in V_h$$

Error equation for the eigenvalue problem

$$a(u - u_h, v) = (\lambda u - \lambda_h u_h, v) \quad \forall v \in V_h$$

Finite element space

- $\mathcal{T}$ conforming and regular triangulation of Ω with regularity constant $\kappa_{\mathcal{T}} = \max_{K\in\mathcal{T}} \frac{\mathrm{diam}(K)}{\rho_K}$.
- $V_{\mathcal{T}}$ finite element space

$$V_{\mathcal{T}} = \{v \in H^1_0(\Omega) : v|_K \in \mathcal{P}_\ell(K) \quad \forall K \in \mathcal{T}\}$$

- If $\mathcal{T}_\star$ is a refinement of $\mathcal{T}$ then $V_{\mathcal{T}} \subset V_{\mathcal{T}_\star}$.
- Given $F \in \mathcal{F}$, we define $\omega_{\mathcal{T}}(F)$ as the union of the two elements in $\mathcal{T}$ sharing F.
- For $K \in \mathcal{T}$, $\mathcal{N}(K) = \{K' \in \mathcal{T} : K' \cup K \neq \emptyset\}$ and $\omega_{\mathcal{T}}(K) = \cup_{K'\in\mathcal{N}(K)} K'$
- Let $n_d = 3$ for $d = 2$ and $n_d = 6$ for $d = 3$. This number guarantees that after n_d bisections to an element, new nodes appear on each side and in the interior. This holds true for example if we consider the newest-vertex bisection when $d = 2$ and the procedure of Kossaczký when $d = 3$.

Residual type *a posteriori* estimates for eigenproblems

Element residual

For $\mu \in \mathbb{R}$ and $v \in V_{\mathcal{T}}$

$$R(\mu, v)|_K = -\Delta v - \mu v \quad \forall K \in \mathcal{T}$$

Jump residual

Let $\mathcal{F}$ be the set of interior faces of the mesh.
Let $\mathcal{F}_K \subset \mathcal{F}$ be the subset of faces of K.
For each face $F \in \mathcal{F}$, let $\mathbf{n}_F$ be an arbitrary unit normal vector.
Let K_{in} and K_{out} be the two elements sharing $\mathcal{F}$, with $\mathbf{n}_F$ pointing towards K_{in}.

$$J(v)|_F = \nabla(v|_{K_{\mathrm{out}}}) \cdot \mathbf{n}_F - \nabla(v|_{K_{\mathrm{in}}}) \cdot \mathbf{n}_F$$

Local and global error estimator

Local error estimator For $\mu \in \mathbb{R}$ and $v \in V_{\mathcal{T}}$, we set

$$\eta^2_{\mathcal{T}}(\mu, v; K) = h_K^2 \|R(\mu, v)\|_K^2 + h_K \|J(v)\|^2_{\partial K} \quad \forall K \in \mathcal{T}$$

Global error estimator For $\mu \in \mathbb{R}$ and $v \in V_{\mathcal{T}}$, we set

$$\eta^2_{\mathcal{T}}(\mu, v) = \sum_{K \in \mathcal{T}} \eta^2_{\mathcal{T}}(\mu, v; K)$$

Upper bound

Theorem

Let $j \in \mathbb{N}$, and let $u_h^{(j)}$ be an eigenfunction corresponding to the jth eigenvalue $\lambda_h^{(j)}$ of the discrete problem, then if $h_{\mathcal{T}}$ is small enough, there exists an eigenfunction $u^{(j)}$ corresponding to the jth eigenvalue $\lambda^{(j)}$ of the continuous problem such that

$$\|u^{(j)} - u_h^{(j)}\|_V \leq C\eta_{\mathcal{T}}^2(\mu, v)$$

Preliminary results

For brevity, we set $u = u^{(j)}$ and $u_h = u_h^{(j)}$ and $e_h = u - u_h$.

Durán-Padra-Rodríguez 2003

First equality

$$(\lambda u - \lambda_h u_h, e_h) = (\lambda + \lambda_h)\,(1 - (u, u_h)) = \frac{\lambda + \lambda_h}{2}\|e_h\|_0^2$$

Second equality

For $v \in V$ there holds

$$a(e_h, v) - (\lambda u - \lambda_h u_h, v) = \sum_{K \in \mathcal{T}_h} \left((\Delta u_h + \lambda_h u_h, v)_K + \frac{1}{2} \sum_{F_K \in \mathcal{F}_K} (J(u_h), v)_{F_K} \right)$$

Preliminary results (cont'd)

Given any $v \in V$, let $v^I \in V_{\mathcal{T}}$ be such that

$$
\begin{aligned}
\|v - v^I\|_{0,K} &\leq C h_K |v|_{1,\omega_{\mathcal{T}}(K)} \\
\|v - v^I\|_{0,F} &\leq C h_F^{1/2} |v|_{1,\omega_{\mathcal{T}}(F)},
\end{aligned}
$$

One can take, for example, the well-known Clément or the Scott-Zhang interpolant of v

Upper error estimate up to higher order term

Theorem

There exists a constant C, depending only on the regularity of $\mathcal{T}$, such that

$$|e_h|_1 \leq C\eta_{\mathcal{T}}(\lambda_h, u_h) + \left(\frac{\lambda + \lambda_h}{2}\right)^{1/2} \|e_h\|_0$$

Proof Let e_h^I an interpolant of e_h, then

$$|e_h|_1^2 = a(e_h, e_h - e_h^I) + (\lambda u - \lambda_h u_h, e_h^I)$$

Upper error estimate up to higher order term

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Proof Let e_h^I an interpolant of e_h, then

$$|e_h|_1^2 = a(e_h, e_h - e_h^I) + (\lambda u - \lambda_h u_h, e_h^I)$$

$$= \sum_{K \in \mathcal{T}_h} \left((\Delta u_h + \lambda_h u_h, e_h - e_h^I)_K + \frac{1}{2} \sum_{F_K \in \mathcal{F}_K} (J(u_h), e_h - e_h^I)_{F_K} \right) + (\lambda u - \lambda_h u_h, e_h)$$

Upper error estimate up to higher order term

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There exists a constant C, depending only on the regularity of $\mathcal{T}$, such that

$$|e_h|_1 \leq C\eta_{\mathcal{T}}(\lambda_h, u_h) + \left(\frac{\lambda + \lambda_h}{2}\right)^{1/2} \|e_h\|_0$$

Proof Let e_h^I an interpolant of e_h, then

$$\begin{aligned} |e_h|_1^2 &= a(e_h, e_h - e_h^I) + (\lambda u - \lambda_h u_h, e_h^I) \\ &= \sum_{K \in \mathcal{T}_h} \left((\Delta u_h + \lambda_h u_h, e_h - e_h^I)_K + \frac{1}{2} \sum_{F_K \in \mathcal{F}_K} (J(u_h), e_h - e_h^I)_{F_K} \right) + (\lambda u - \lambda_h u_h, e_h) \\ &\leq C\eta_{\mathcal{T}}(\lambda_h, u_h)|e_h|_1 + \frac{\lambda + \lambda_h}{2} \|e_h\|_0^2 \end{aligned}$$

Regularity of the eigenfunctions and a priori estimate

Regularity

There exists $r \in]0,1]$ depending only on Ω such that

$$u \in H^{1+r}(\Omega)$$

for any eigenfunction u.

A priori error estimate

If $h_{\mathcal{T}}$ is small enough, then

$$\|u - u_h\|_0 \leq C h_{\mathcal{T}}^r |u - u_h|_1$$

Hence we get

$$|e_h|_1 \leq C(\eta_{\mathcal{T}}(\lambda_h, u_h) + h_{\mathcal{T}}^r |e_h|_1)$$

Discrete local error bound

Let $K \in \mathcal{T}$ and let $\mathcal{T}_\star$ be obtained from $\mathcal{T}$ by bisecting n_d times each element of $\mathcal{N}_\mathcal{T}(K)$. Let $\lambda_\mathcal{T}$ and $u_\mathcal{T}$ be a solution to (EIG_h). Let $\mathbb{W}_h$ be a subspace of $H^1_0(\Omega)$ such that $V_{\mathcal{T}_\star} \subset \mathbb{W}_h$. If $\mu \in \mathbb{R}$ and $w \in \mathbb{W}_h$ satisfy

$$\begin{cases} a(w,v) = \mu(w,v) \quad \forall v \in \mathbb{W}_h \\ \|w\|_0 = 1 \end{cases}$$

then

$$\begin{aligned} \eta_\mathcal{T}(\lambda_\mathcal{T}, u_\mathcal{T}; K) \leq & C\Big(\|\nabla(w-u_\mathcal{T})\|_{\omega_\mathcal{T}(K)} + h_K \|\mu w - \lambda_\mathcal{T} u_\mathcal{T}\|_{\omega_\mathcal{T}(K)} \Big) \\ & C\Big(\|\nabla(w-u_\mathcal{T})\|_{\omega_\mathcal{T}(K)} + h_K \|\mu w\|_{\omega_\mathcal{T}(K)} + h_K \lambda_\mathcal{T} \|u_\mathcal{T}\|_{\omega_\mathcal{T}(K)} \Big) \end{aligned}$$

where, for every $K' \in \mathcal{N}_\mathcal{T}(K)$, $\overline{R}|_{K'}$ is the $L^2(K')$-projection of $R = R(\lambda_\mathcal{T}, u_\mathcal{T})$ onto $\mathcal{P}_{\ell-1}$, and for every side $F \subset \partial K$, $\overline{J}|_F$ is the $L^2(F)$-projection of $J = J(u_\mathcal{T})$ onto $\mathcal{P}_{\ell-1}$.

Adaptive loop

Let λ be the jth eigenvalue and u an eigenfunction in $E(\lambda)$

Algorithm
Start with an initial mesh $\mathcal{T}_0$

1. $(\lambda_k, u_k) = \mathrm{SOLVE}(V_k)$
2. $\{\eta_k(K)\}_{K\in\mathcal{T}_k} = \mathrm{ESTIMATE}(\lambda_k, u_k, \mathcal{T}_k)$
3. $\mathcal{M} = \mathrm{MARK}(\{\eta_k(K)\}_{K\in\mathcal{T}_k}, \mathcal{T}_k)$
4. $\mathcal{T}_{k+1} = \mathrm{REFINE}(\mathcal{T}_k, \mathcal{M}_k)$, increment k

Convergence to a limiting pair

Let $V_\infty = \overline{\cup V_k}^{H^1_0(\Omega)}$

Theorem

There exist λ_∞ and $u_\infty \in V_\infty$ such that

$$\begin{cases} a(u_\infty, v) = \lambda_\infty (u_\infty, v) \quad \forall v \in V_\infty \\ \|u_\infty\|_0 = 1 \end{cases}$$

Moreover

$$\lambda_\infty = \lim_{k\to\infty} \lambda_k$$

and there exists a subsequence $\{u_{k_m}\}_{m\in\mathbb{N}}$ of $\{u_k\}_{k\in\mathbb{N}}$ such that

$$u_{k_m} \to u_\infty \quad \text{in } H^1_0(\Omega)$$

1 $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{T}_k$, the *min-max* principle implies that $\{\lambda_k\}_{k\in\mathbb{N}}$ is a decreasing sequence bounded below by λ. There exists $\lambda_\infty > 0$ such that

$$\lambda_k \searrow \lambda_\infty$$

2 $|u_k|_1^2 = \lambda_k \to \lambda_\infty$ is bounded in V_∞

3 A subsequence $\{u_{k_m}\}_{m\in\mathbb{N}}$ converges weakly in V_∞ to $u_\infty \in V_\infty$, so $u_{k_m} \rightharpoonup u_\infty$ in $H_0^1(\Omega)$

4 By compactness, we have strong convergence $u_{k_m} \to u_\infty$ in $L^2(\Omega)$

5 If $k_0 \in \mathbb{N}$ and $k_m \geq k_0$, for all $v_{k_0} \in V_{k_0}$ $a(u_{k_m}, v_{k_0}) = \lambda_{k_m}(u_{k_m}, v_{k_0})$

6 When $m \to \infty$ $a(u_\infty, v_{k_0}) = \lambda_\infty(u_\infty, v_{k_0})$

7 $k_0 \in \mathbb{N}$ and $v_{k_0} \in V_{k_0}$ are arbitrary, so that

$$a(u_\infty, v) = \lambda_\infty(u_\infty, v) \quad \forall v \in V_\infty$$

8 From strong convergence and $\|u_{k_m}\|_0 = 1$ imply $\|u_\infty\|_0 = 1$ and $\|u_\infty\|_1 = \lambda_\infty$

9 $|u_{k_m}|_1^2 = \lambda_{k_m} \to \lambda_\infty$, so that $|u_{k_m}|_1 \to |u_\infty|_1$. That implies

$$u_{k_m} \to u_\infty \quad \text{in } H_0^1(\Omega).$$

Convergence of estimators

Set:

- $\mathcal{T}_k^0 = \{K \in \mathcal{T}_k : K'$ is refined at least n_d times for all $K' \in \mathcal{N}(K)\}$
- $\mathcal{T}_k^+ = \{K \in \mathcal{T}_k : K'$ is never refined for all $K' \in \mathcal{N}(K)\}$
- $\mathcal{T}_k^\star = \mathcal{T}_k \setminus (\mathcal{T}_k^0 \cup \mathcal{T}_k^\star)$

Theorem

Then the contributions of $\mathcal{T}_k^0$, $\mathcal{T}_k^\star$ and $\mathcal{T}_k^+$ to the estimator vanish in the limit, i.e

$$\lim_{k\to\infty} \eta_k(\mathcal{T}_k^0) = 0$$
$$\lim_{k\to\infty} \eta_k(\mathcal{T}_k^\star) = 0$$
$$\lim_{k\to\infty} \eta_k(\mathcal{T}_k^+) = 0$$

The limiting pair is an eigenpair

Theorem

The limiting pair $(\lambda_\infty, u_\infty) \in \mathbb{R} \times V_\infty$ is an eigenpair of the continuous problem, that is

$$\begin{cases} a(u_\infty, v) = \lambda_\infty(u_\infty, v) \quad \forall v \in H^1_0(\Omega) \\ \|u_\infty\|_0 = 1 \end{cases}$$

Let $v \in H^1_0(\Omega)$. With some computation we get:

$$|a(u_\infty, v) - \lambda_\infty(u_\infty, v)| \le |a(u_k, v - v_k) - \lambda_k(u_k, v - v_k)| + |(\lambda_k u_k - \lambda_\infty u_\infty, v)| + a(u_\infty - u_k, v)|$$

Using the same steps as in the proof of reliability we get

$$|a(u_k, v - v_k) - \lambda_k(u_k, v - v_k)| \le C\eta_(\mathcal{T}_k)|v|_1$$

so that

$$|a(u_\infty, v) - \lambda_\infty(u_\infty, v)| \le C((1 + \lambda_0)\|u_k - u_\infty\|_1 + |\lambda - \lambda_\infty|\|u_\infty\|_0 + \eta_(\mathcal{T}_k)\|v\|_1$$

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The limiting pair $(\lambda_\infty, u_\infty) \in \mathbb{R} \times V_\infty$ is an eigenpair of the continuous problem, that is

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Remark This proof does not require that the initial mesh is sufficiently fine.

Main result

Let $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}}$ be the sequence of discrete eigenpairs obtained through the adaptive loop. Then, there exists an eigenvalue λ of the continuous problem such that

$$\lim_{k\to\infty} \lambda_k = \lambda, \qquad \lim_{k\to\infty} \delta(u_k, E(\lambda)) = 0$$

Open question: If λ_k was chosen as the jth eigenvalue of the discrete problem over $\mathcal{T}_k$, is it true that $\{\lambda_k\}_{k\in\mathbb{N}}$ converges to the jth eigenvalue of the continuous problem?

Non degeneracy assumption

We will say that problem satisfies the *Non-Degeneracy Assumption* if whenever u is an eigenfunction, there is no nonempty open subset $\mathcal{O}$ of Ω such that $u|_{\mathcal{O}} \in \mathcal{P}_(\mathcal{O})$.

Theorem

Let $\{(\lambda_k, u_k)\}_{k\in\mathbb{N}}$ be the sequence of discrete eigenpairs obtained through the adaptive loop and λ denote the jth eigenvalue of the continuous problem. Then

$$\lim_{k\to\infty} \lambda_k = \lambda, \qquad \lim_{k\to\infty} \delta(u_k, E(\lambda)) = 0$$

Outline

References

A posteriori estimates for eigenvalue problem

AFEM for eigenvalue problems in mixed form

Numerical results

Laplace eigenvalue problem in mixed form

Problem

Find $\lambda \in \mathbb{R}$ and $u \in L^2(\Omega)$ with $\|u\|_0 = 1$ such that for some $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div};\Omega)$ it holds that

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\, \boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div};\Omega) \\ (\mathrm{div}\, \boldsymbol{\sigma}, v) = -\lambda(u, v) & \forall v \in L^2(\Omega) \end{cases}$$

Laplace eigenvalue problem in mixed form

Problem

Find $\lambda \in \mathbb{R}$ and $u \in L^2(\Omega)$ with $\|u\|_0 = 1$ such that for some $\boldsymbol{\sigma} \in \mathbf{H}(\mathrm{div};\Omega)$ it holds that

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}(\mathrm{div};\Omega) \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) = -\lambda(u, v) & \forall v \in L^2(\Omega) \end{cases}$$

$\boldsymbol{\Sigma}_h \subset \mathbf{H}(\mathrm{div};\Omega)$, $U_h \subset L^2(\Omega)$ mixed FE with $\mathrm{div}(\boldsymbol{\Sigma}_h) = U_h$ (Raviart-Thomas, BDM)

Discrete problem

Find $\lambda_h \in \mathbb{R}$ and $u_h \in U_h$ with $\|u_h\|_0 = 1$ such that for some $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h$ it holds that

$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u_h) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \\ (\mathrm{div}\,\boldsymbol{\sigma}_h, v) = -\lambda_h(u_h, v) & \forall v \in U_h \end{cases}$$

We set $\mathbb{K}_h = \{\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h : (\mathrm{div}\,\boldsymbol{\tau}, v) = 0\ \forall v \in U_h\}$

Eigenvalue clusters and error indicators

Eigenvalue cluster

For a cluster of eigenvalues $\lambda_{n+1}, \dots, \lambda_{n+\mathrm{N}}$ of length $\mathrm{N} \in \mathbb{N}$, we define the index set $J = \{n+1, \dots, n+\mathrm{N}\}$ and the spaces

$$W = \operatorname{span}\{u_j \mid j \in J\} \quad \text{and} \quad W_{\mathcal{T}_h} = W_h = \operatorname{span}\{u_{h,j} \mid j \in J\}.$$

Local error indicator

Let $(\boldsymbol{\sigma}_{h,j}, u_{h,j}) \in \mathbb{R} \times U_h$ be a discrete eigensolution computed on the mesh $\mathcal{T}_h$. Then for all $K \in \mathcal{T}_h$ we define

$$\eta_{h,j}(K)^2 = \|h_K(\boldsymbol{\sigma}_{h,j} - \mathbf{grad}\, u_{h,j})\|_K^2 + \|h_K\, \mathbf{curl}\, \boldsymbol{\sigma}_{h,j}\|_K^2 + \sum_{F \in \mathcal{F}_K} h_F \|[\![\boldsymbol{\sigma}_{h,j}]\!]_F \cdot t_F\|_F^2,$$

Given a set $\mathcal{M} \subset \mathcal{T}_h$ we define

$$\eta_{h,j}(\mathcal{M})^2 = \sum_{K \in \mathcal{M}} \eta_{h,j}(K)^2.$$

Adaptive algorithm

We describe how the algorithms runs from level ℓ to $\ell+1$.
Solve Given a mesh $\mathcal{T}_\ell$, compute the eigensolutions in the cluster $(\lambda_{\ell,j}, \sigma_{\ell,j}, u_{\ell,j})$ for $j \in J$.
Estimate Compute the local error estimator for the eigenfunctions in the cluster $\{\eta_{\ell,j}(T)\}_{T\in\mathcal{T}_\ell}$ $(j \in J)$.
Mark Use the Dörfler marking strategy: given a bulk parameter $\theta \in (0,1]$, find a minimal subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ such that

$$\theta \sum_{j\in J} \eta_{\ell,j}(\mathcal{T}_\ell)^2 \leq \sum_{j\in J} \eta_{\ell,j}(\mathcal{M}_\ell)^2.$$

The elements belonging to $\mathcal{M}_\ell$ are marked for refinement.
Refine $\mathcal{T}_{\ell+1}$ is generated, as the smallest admissible refinement of $\mathcal{T}_\ell$ satisfying $\mathcal{M}_\ell \cap \mathcal{T}_{\ell+1} = \emptyset$ by using refinement rules.

Optimal convergence

A nonlinear approximation class

For any $m \in \mathbb{N}$, we denote by

$$\mathbb{T}(m) = \{\mathcal{T} \in \mathbb{T} \mid \operatorname{card}(\mathcal{T}) - \operatorname{card}(\mathcal{T}_0) \leq m\}$$

the set of admissible triangulations in $\mathbb{T}$ whose cardinality differs from that of $\mathcal{T}_0$ by m or less.

The best algebraic convergence rate $s \in (0, +\infty)$ obtained by any admissible mesh in $\mathbb{T}$ is characterized in terms of the following seminorm

$$|W|_{\mathcal{A}_s} = \sup_{m \in \mathbb{N}} m^s \inf_{\mathcal{T} \in \mathbb{T}(m)} \delta(W, W_{\mathcal{T}}).$$

In particular, $|W|_{\mathcal{A}_s} < \infty$ if the rate of convergence $\delta(W, W_{\mathcal{T}}) = O(m^{-s})$ holds true for the optimal triangulations $\mathcal{T}$ in $\mathbb{T}(m)$.

Optimal convergence (cont'd)

Theorem

Provided the initial mesh-size and the bulk parameter θ are small enough, if for the eigenvalue cluster W it holds $|W|_{\mathcal{A}_s} < \infty$, then the sequence of discrete clusters W_ℓ computed on the mesh $\mathcal{T}_\ell$ satisfies the optimal estimate

$$\delta(W, W_\ell)(\mathrm{card}(\mathcal{T}_\ell) - \mathrm{card}(\mathcal{T}_0))^s \leq C|W|_{\mathcal{A}_s}.$$

Let J denote the set of indices corresponding to the eigenvalues in the cluster W. Then

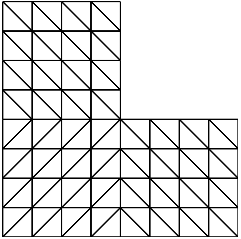
$$\sup_{i\in J} \inf_{j\in J} |\lambda_i - \lambda_{\ell,j}| \leq C\delta(W, W_\ell)^2.$$

- Superconvergence for the eigenvalue problem
- Efficiency and discrete reliability
- Quasi-orthogonality
- Contraction property

Outline

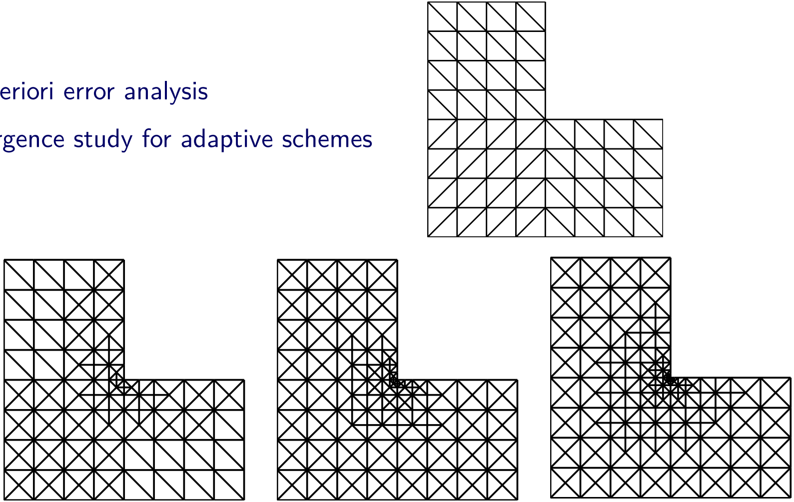
Some comments on adaptive schemes

- A posteriori error analysis
- Convergence study for adaptive schemes



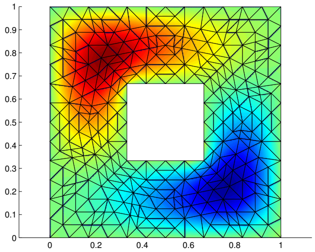
Some comments on adaptive schemes

- A posteriori error analysis
- Convergence study for adaptive schemes

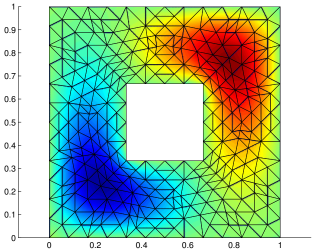


Multiple eigenvalues: the square ring

λ_2



$\lambda_3 = \lambda_2$



Question

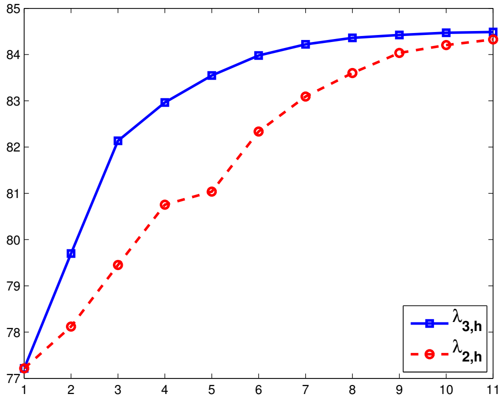
What is the best adaptive strategy for the approximation of the multiple eigenvalue?

1. Indicator based on $(\lambda_{h,2}, u_{h,2})$
2. Indicator based on $(\lambda_{h,3}, u_{h,3})$
3. Indicator based on both $(\lambda_{h,2}, u_{h,2})$ and $(\lambda_{h,3}, u_{h,3})$

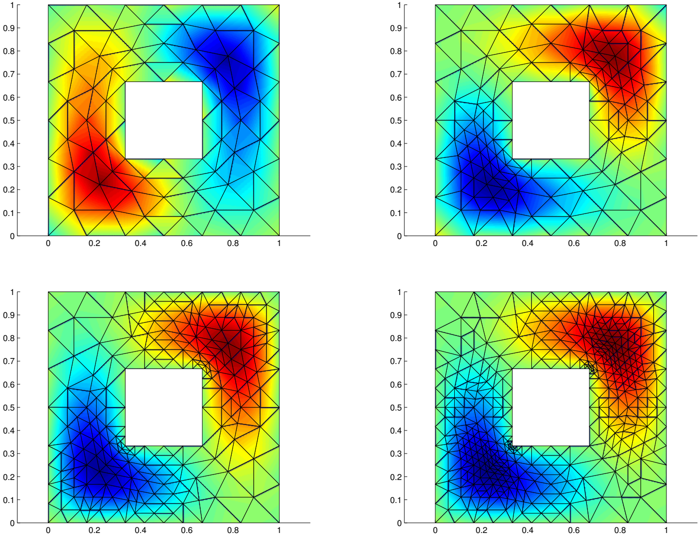
Refinement based on $\lambda_{h,3}$

B.–Durán–Gardini–Gastaldi 2015

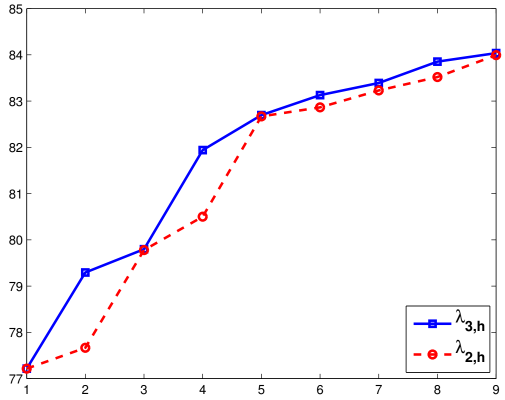
Remark: here we are using a nonconforming discretization which provides eigenvalue approximation from below



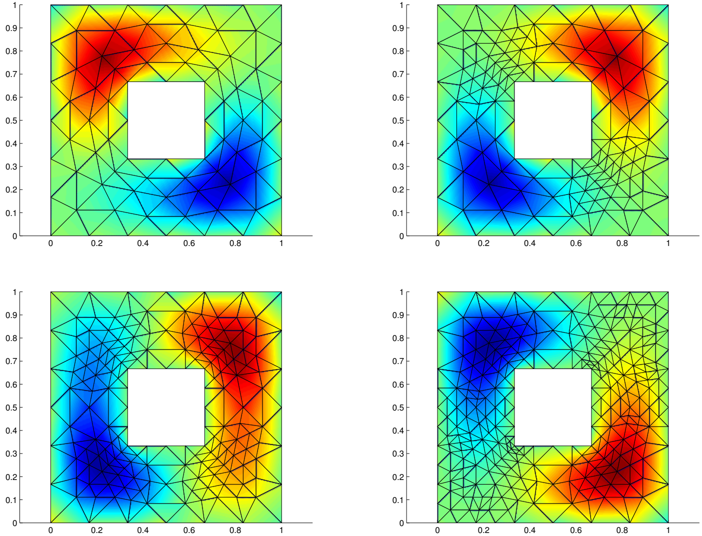
Refinement based on $\lambda_{h,3}$ (eigenfunction $u_{h,3}$)



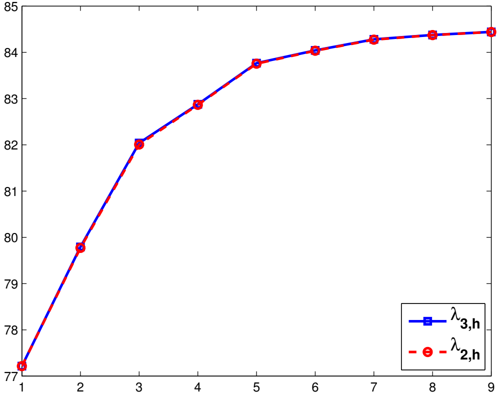
Refinement based on $\lambda_{h,2}$



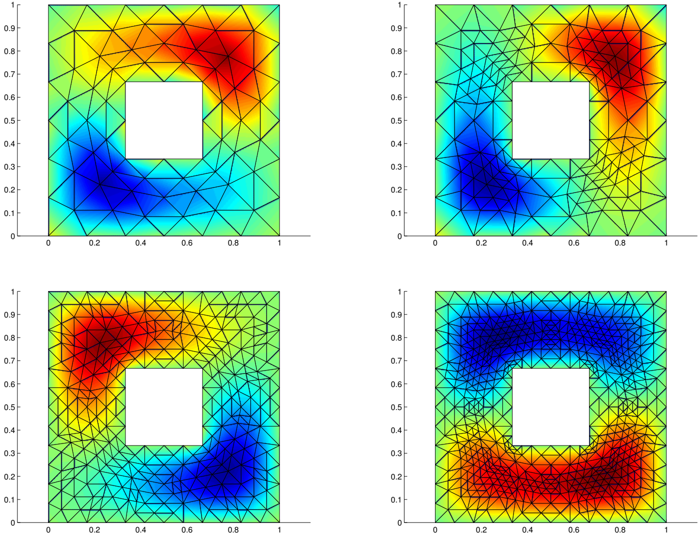
Refinement based on $\lambda_{h,2}$ (eigenfunction $u_{h,2}$)



Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenvalues)



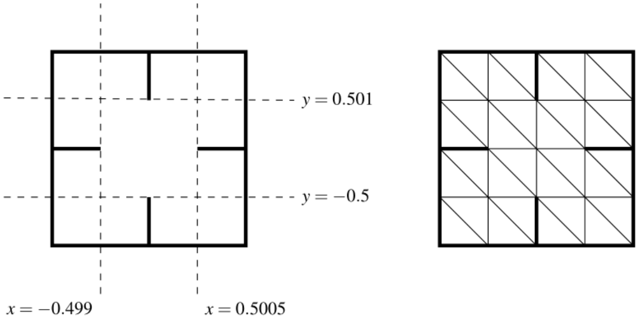
Refinement based on $\lambda_{h,2}$ and $\lambda_{h,3}$ (eigenfunction $u_{h,2}$)



Cluster of eigenvalues

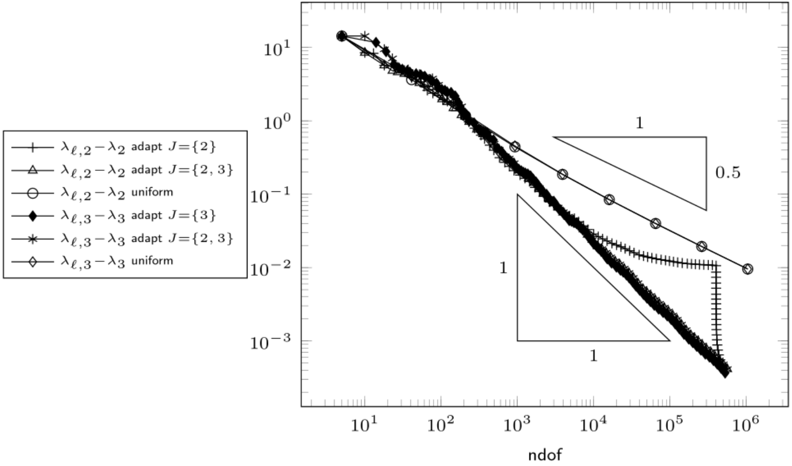
Gallistl '14

A slightly non-symmetric domain



Now $\lambda_2 < \lambda_3$ but they are very close to each other

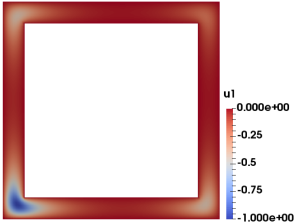
Non-symmetric slit domain



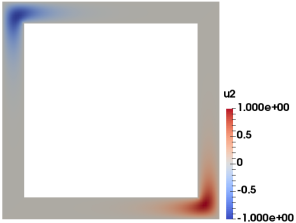
A slightly non-symmetric domain

A square ring for which the first four modes are the following ones (computed on an adapted mesh with 4,122,416 dof's)

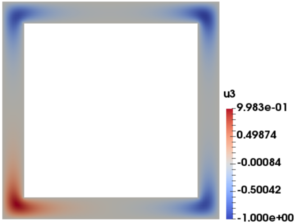
916.745



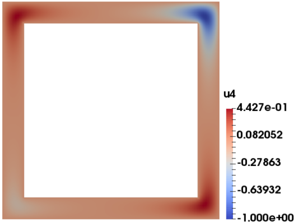
917.008



917.010

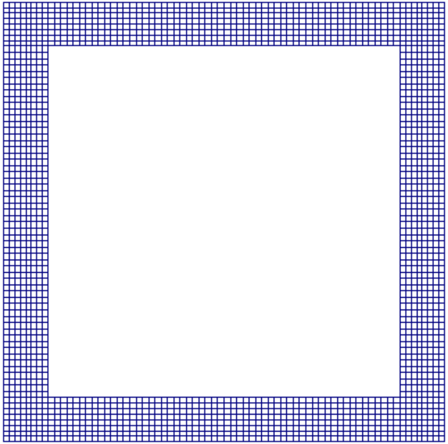


917.274



Approximation of first frequency

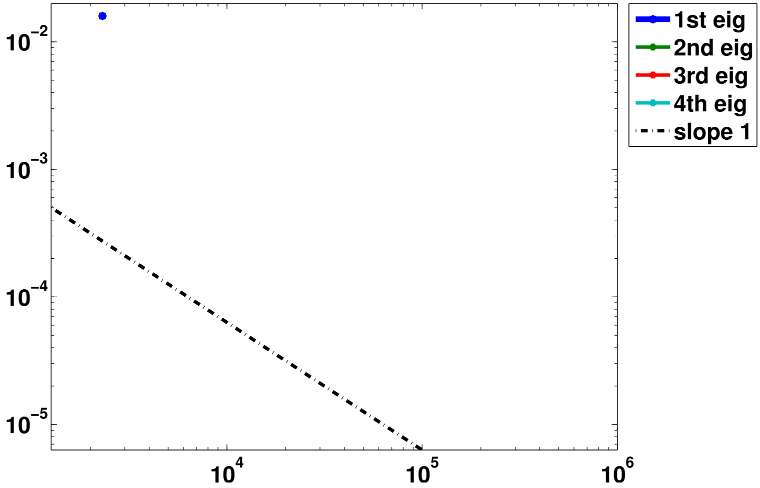
Initial mesh



At each refinement level we compute the first four eigenmodes and drive the adaptive strategy according to the error indicator related to the first eigenmode

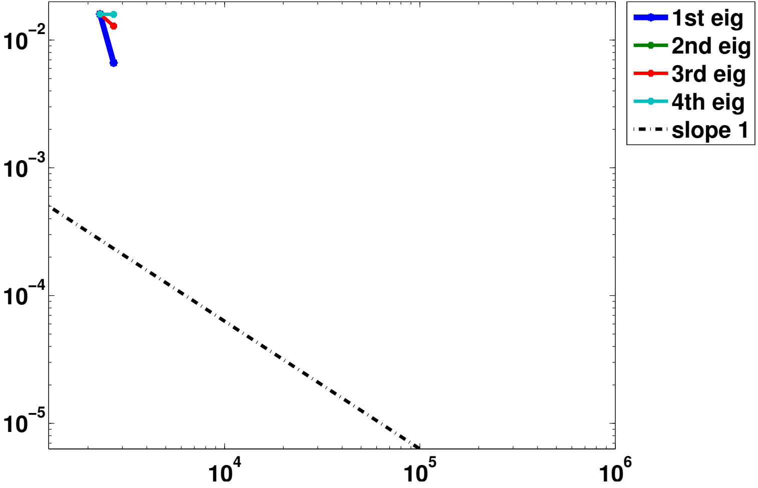
Approximation of first frequency

Bulk parameter=0.3, Refinement level=0 (initial mesh)



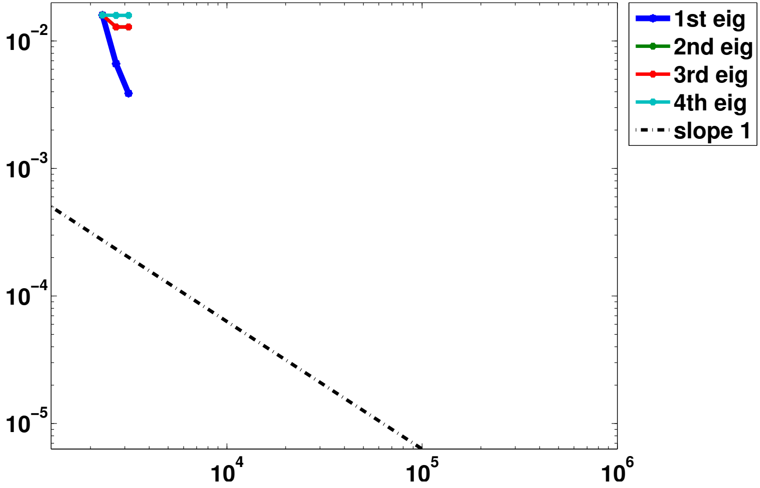
Approximation of first frequency

Bulk parameter=0.3, Refinement level=1



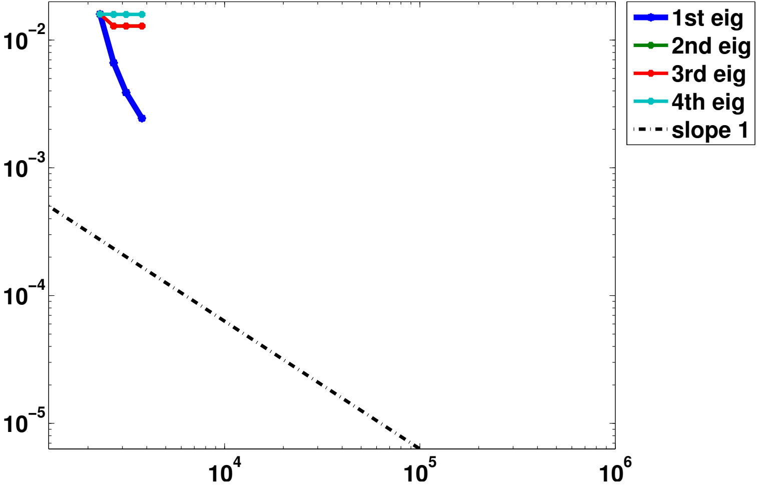
Approximation of first frequency

Bulk parameter=0.3, Refinement level=2



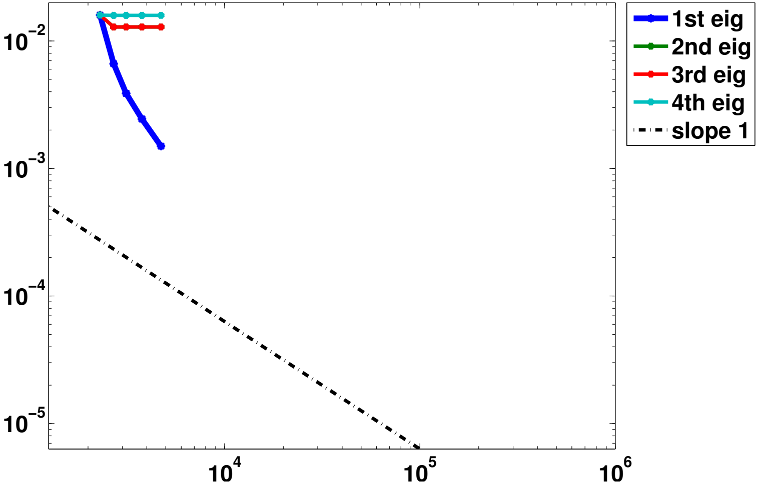
Approximation of first frequency

Bulk parameter=0.3, Refinement level=3



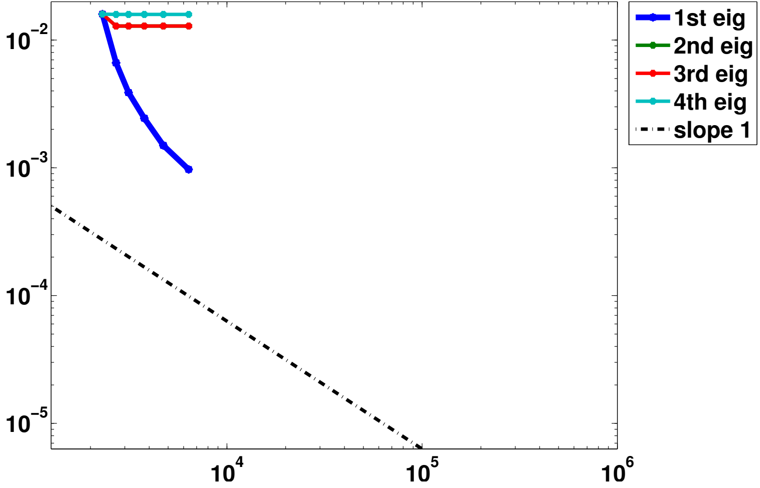
Approximation of first frequency

Bulk parameter=0.3, Refinement level=4



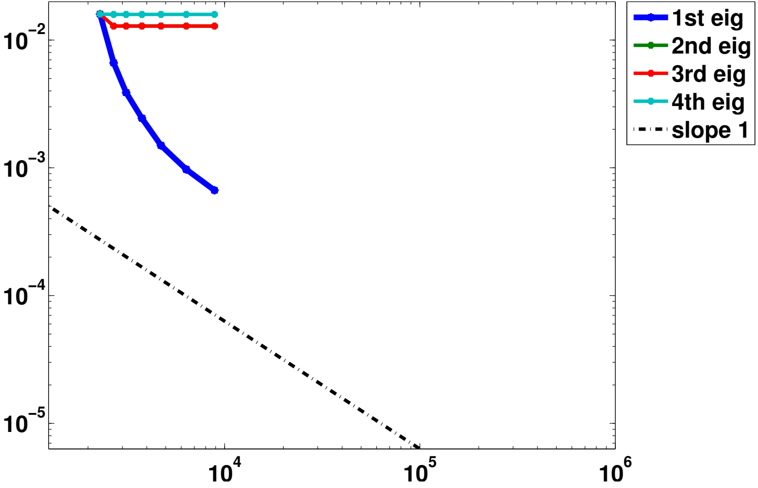
Approximation of first frequency

Bulk parameter=0.3, Refinement level=5



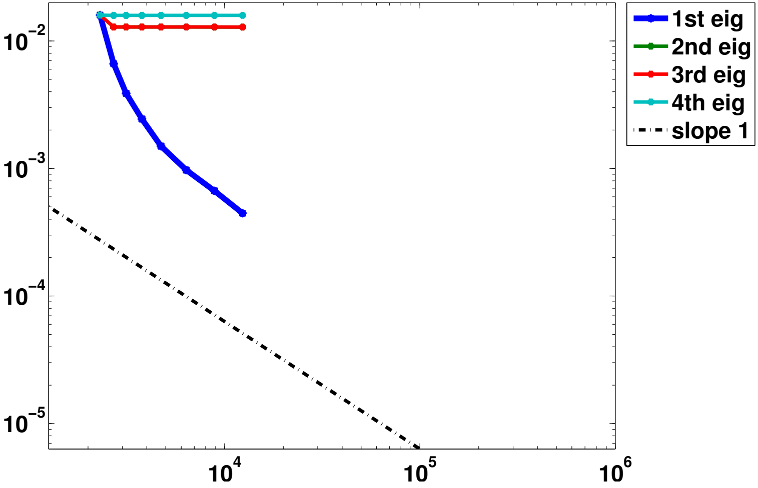
Approximation of first frequency

Bulk parameter=0.3, Refinement level=6



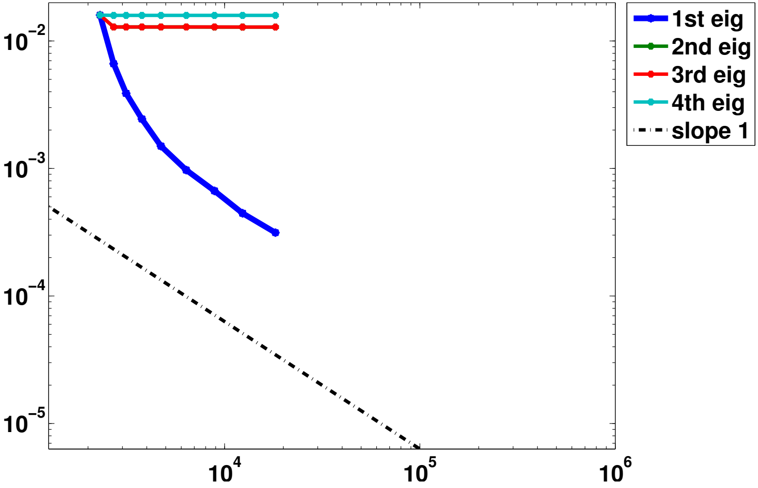
Approximation of first frequency

Bulk parameter=0.3, Refinement level=7



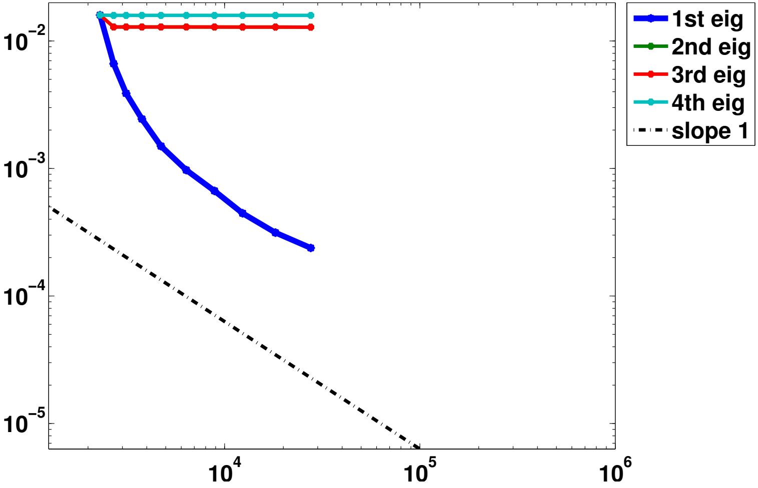
Approximation of first frequency

Bulk parameter=0.3, Refinement level=8



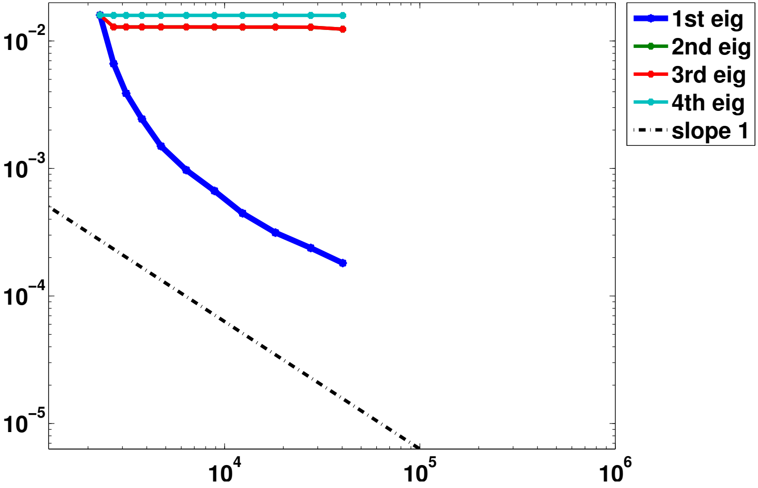
Approximation of first frequency

Bulk parameter=0.3, Refinement level=9



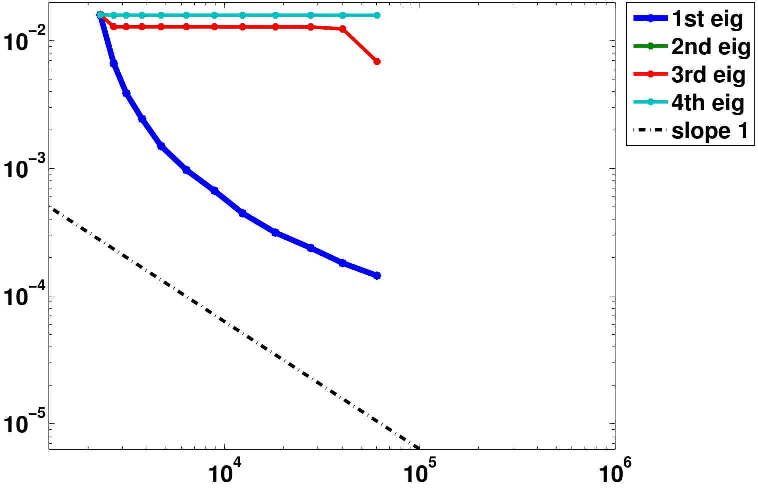
Approximation of first frequency

Bulk parameter=0.3, Refinement level=10



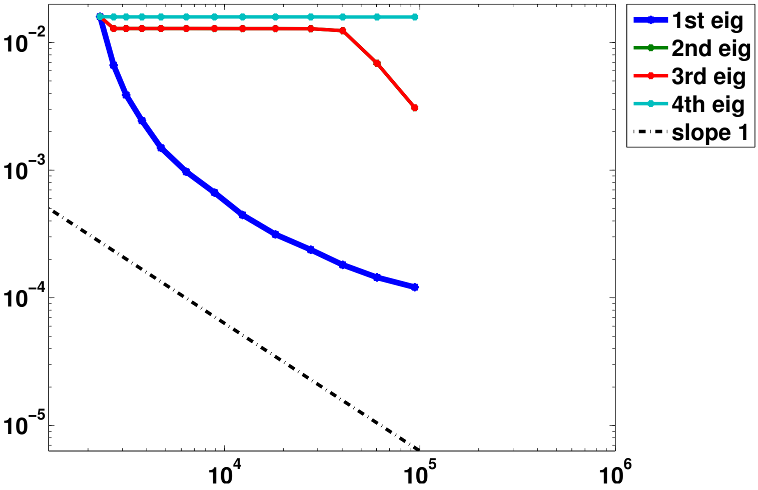
Approximation of first frequency

Bulk parameter=0.3, Refinement level=11



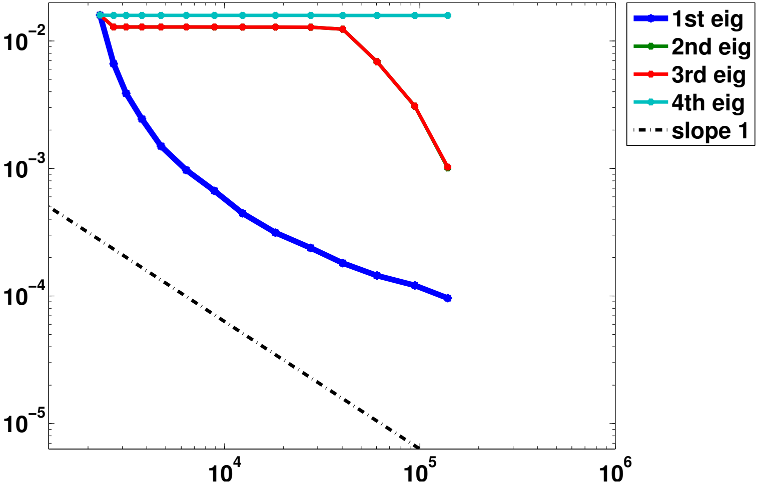
Approximation of first frequency

Bulk parameter=0.3, Refinement level=12



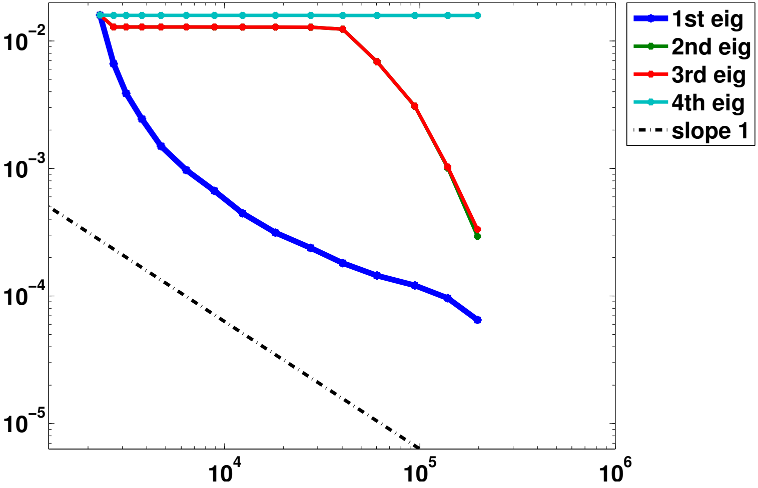
Approximation of first frequency

Bulk parameter=0.3, Refinement level=13



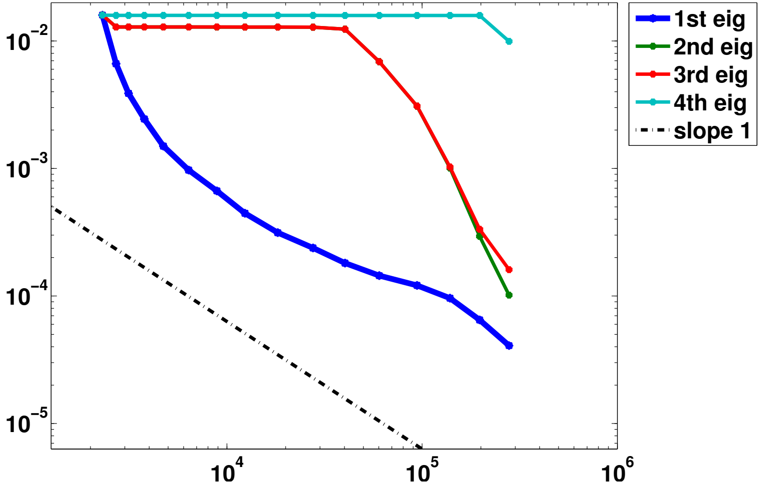
Approximation of first frequency

Bulk parameter=0.3, Refinement level=14



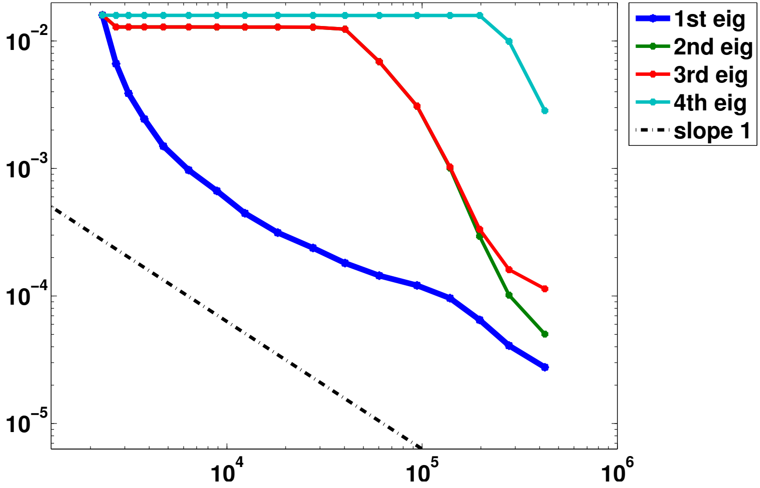
Approximation of first frequency

Bulk parameter=0.3, Refinement level=15



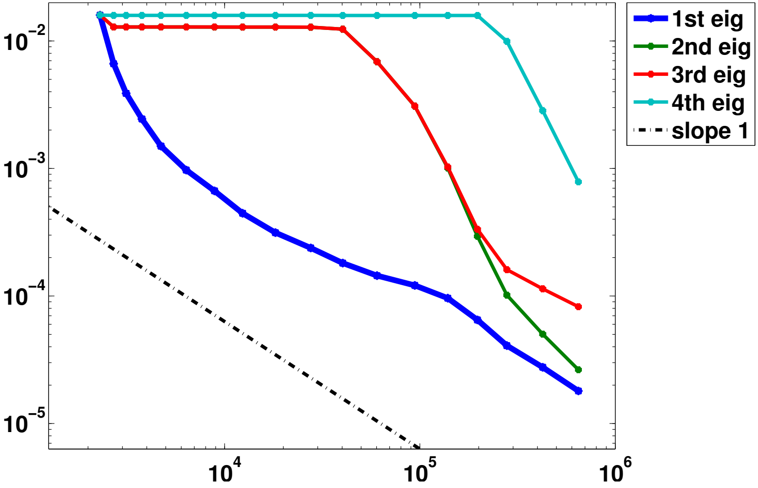
Approximation of first frequency

Bulk parameter=0.3, Refinement level=16



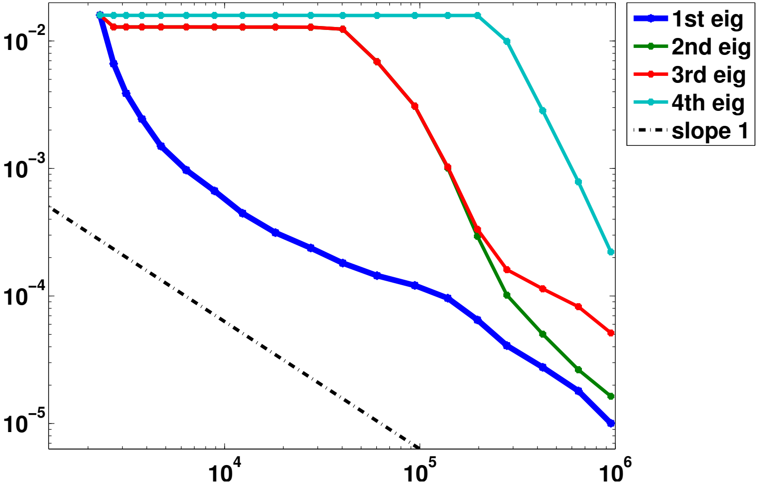
Approximation of first frequency

Bulk parameter=0.3, Refinement level=17



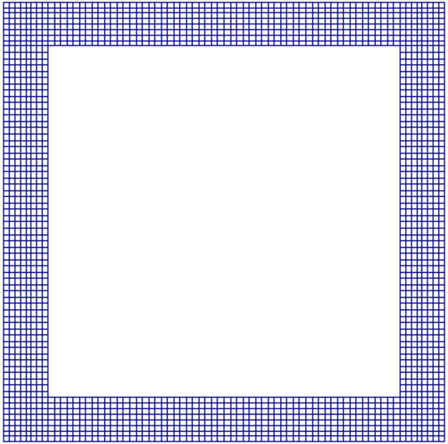
Approximation of first frequency

Bulk parameter=0.3, Refinement level=18



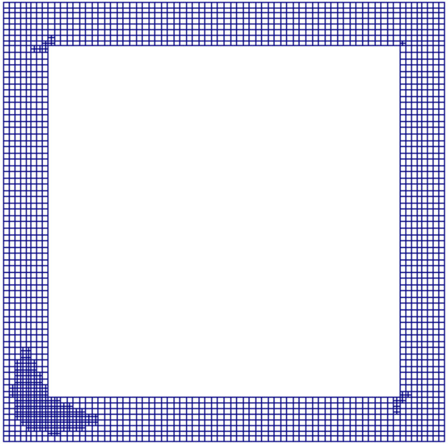
Approximation of first frequency: underlying mesh

Initial mesh



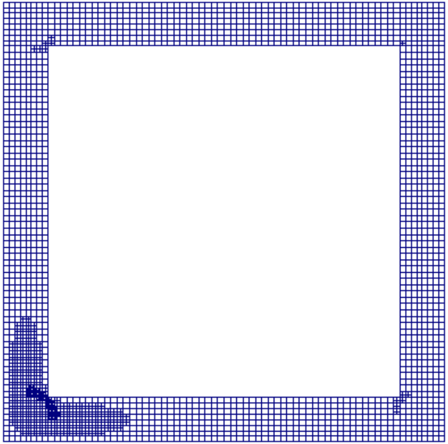
Approximation of first frequency: underlying mesh

Refinement level=1



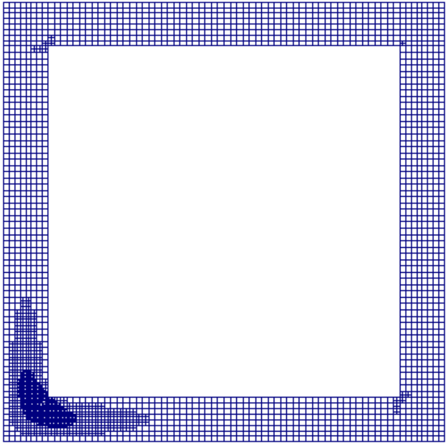
Approximation of first frequency: underlying mesh

Refinement level=2



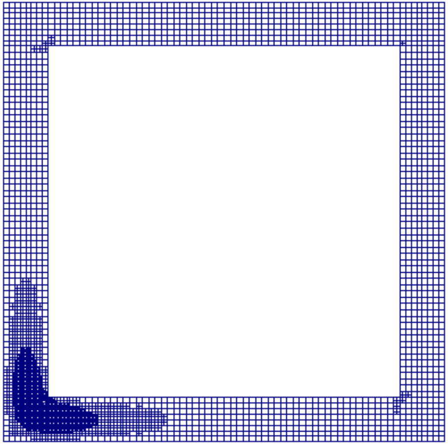
Approximation of first frequency: underlying mesh

Refinement level=3



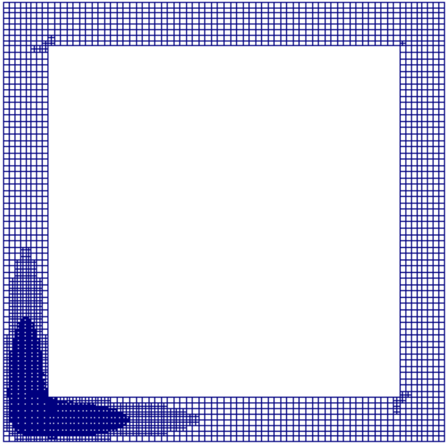
Approximation of first frequency: underlying mesh

Refinement level=4



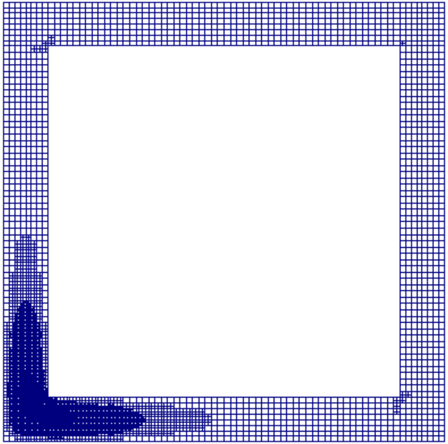
Approximation of first frequency: underlying mesh

Refinement level=5



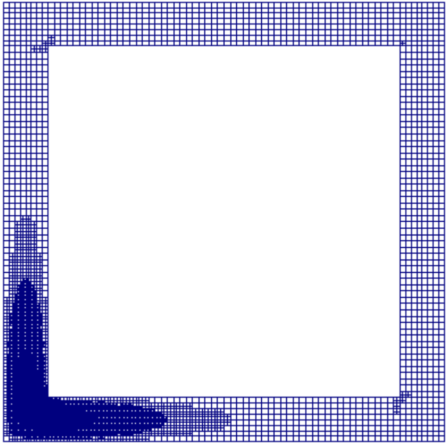
Approximation of first frequency: underlying mesh

Refinement level=6



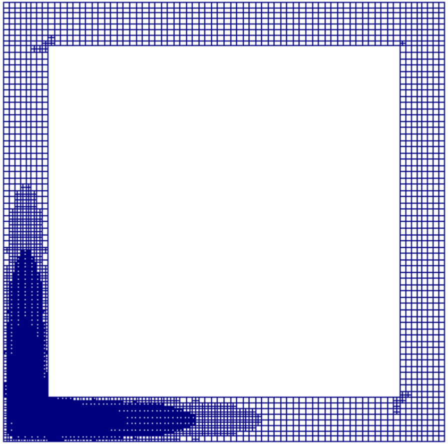
Approximation of first frequency: underlying mesh

Refinement level=7



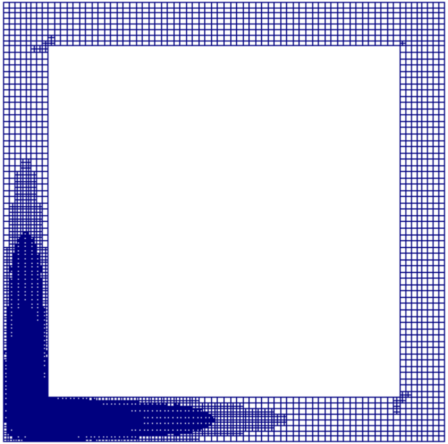
Approximation of first frequency: underlying mesh

Refinement level=8



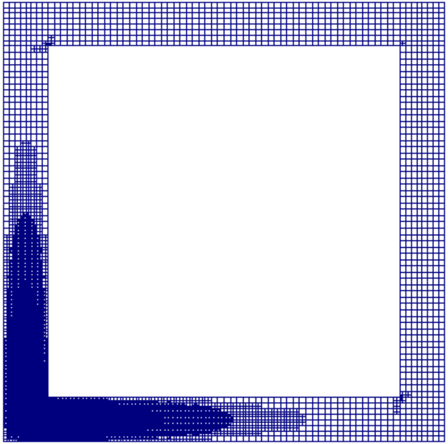
Approximation of first frequency: underlying mesh

Refinement level=9



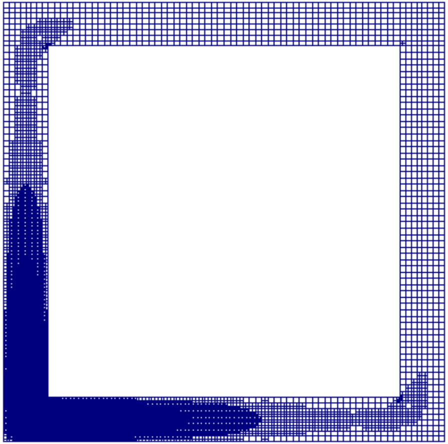
Approximation of first frequency: underlying mesh

Refinement level=10



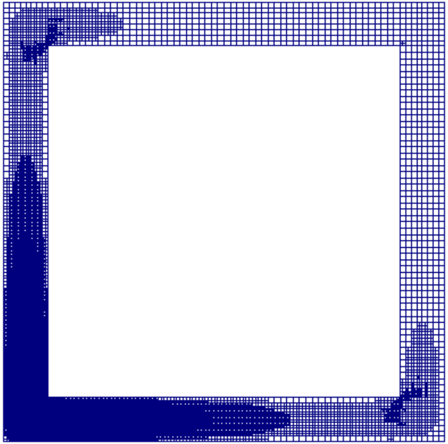
Approximation of first frequency: underlying mesh

Refinement level=11



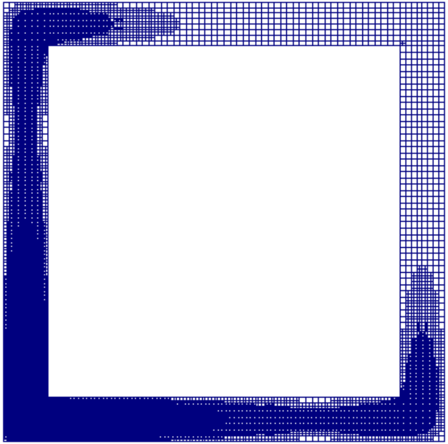
Approximation of first frequency: underlying mesh

Refinement level=12



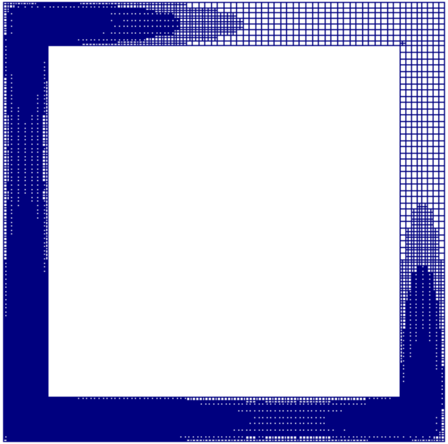
Approximation of first frequency: underlying mesh

Refinement level=13



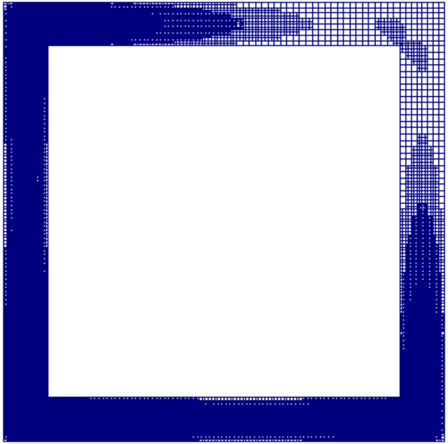
Approximation of first frequency: underlying mesh

Refinement level=14



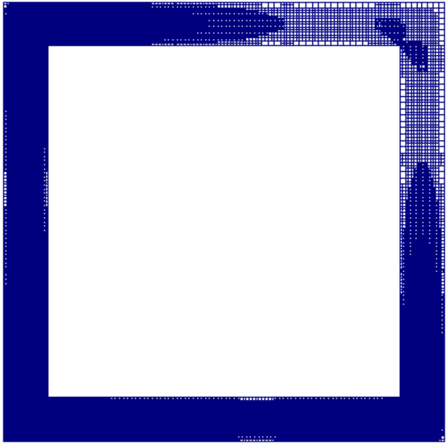
Approximation of first frequency: underlying mesh

Refinement level=15



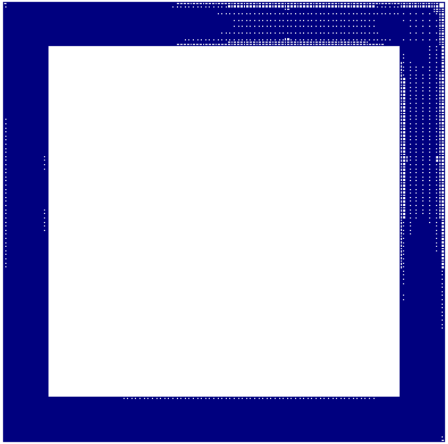
Approximation of first frequency: underlying mesh

Refinement level=16



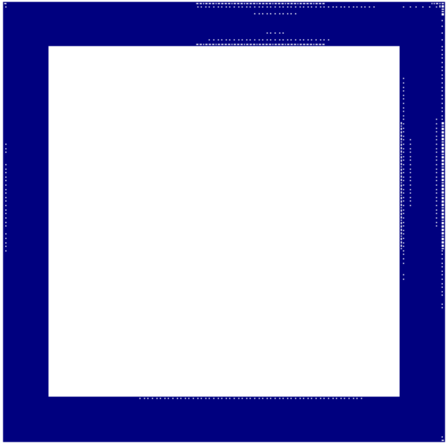
Approximation of first frequency: underlying mesh

Refinement level=17



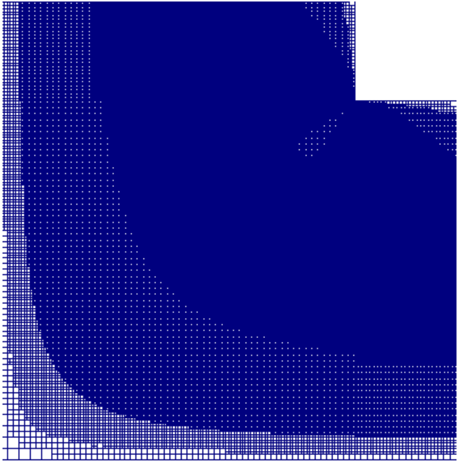
Approximation of first frequency: underlying mesh

Refinement level=18

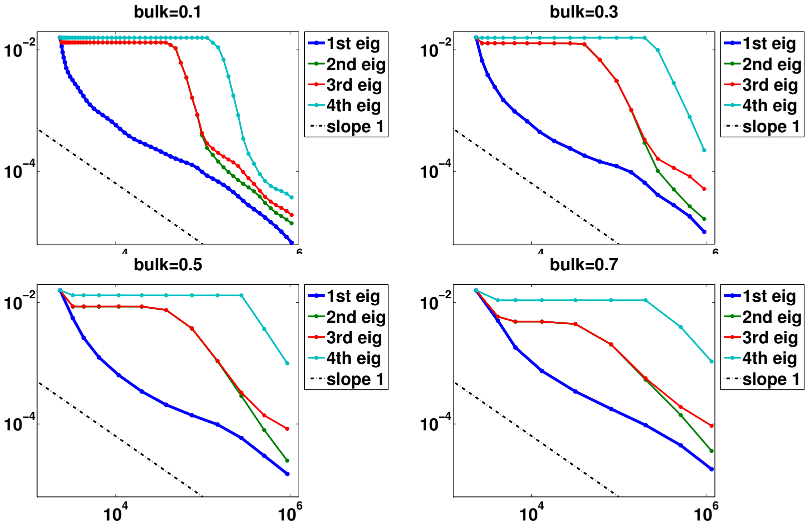


Approximation of first frequency: underlying mesh

Detail of the last mesh

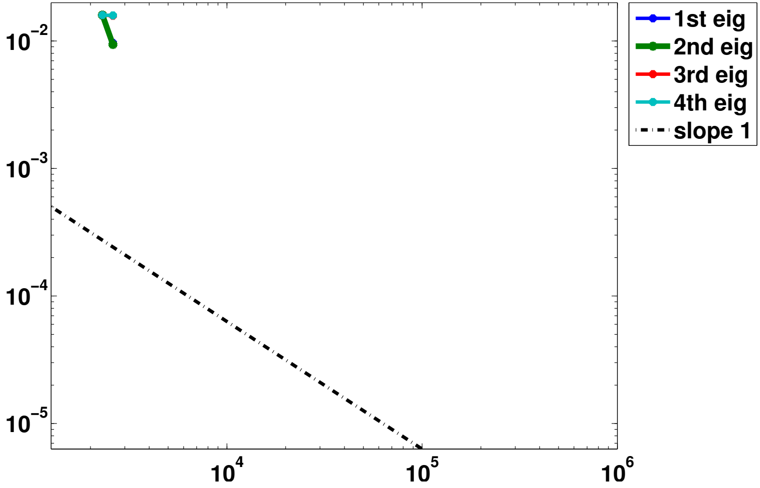


Changing the bulk parameter doesn't help



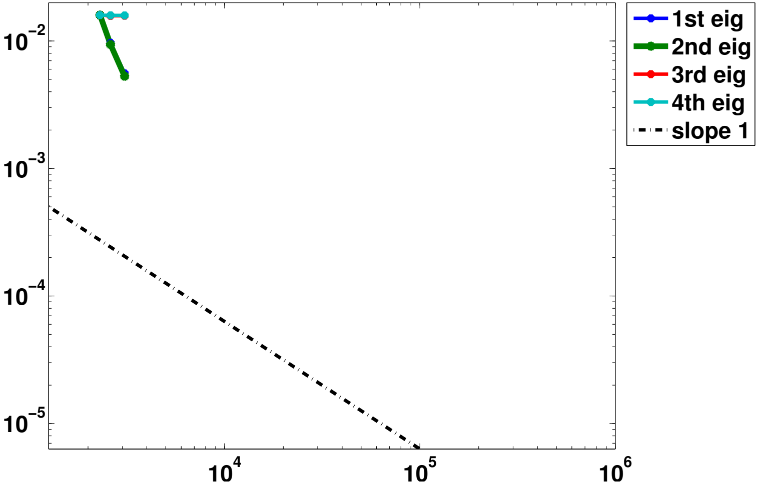
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=1



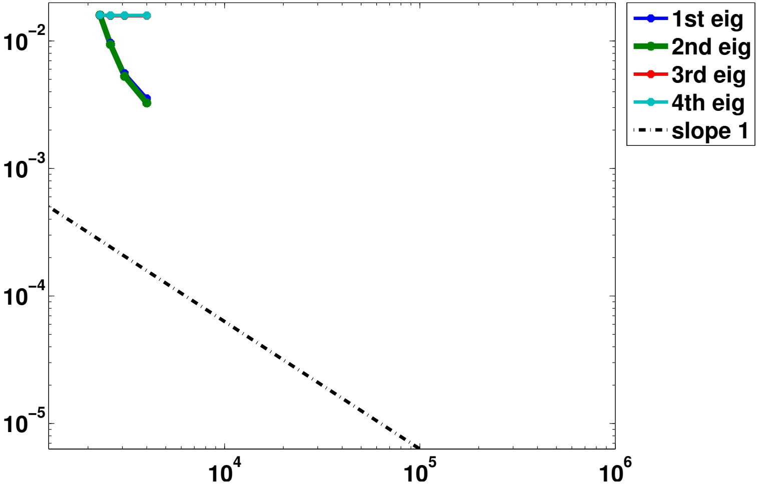
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=2



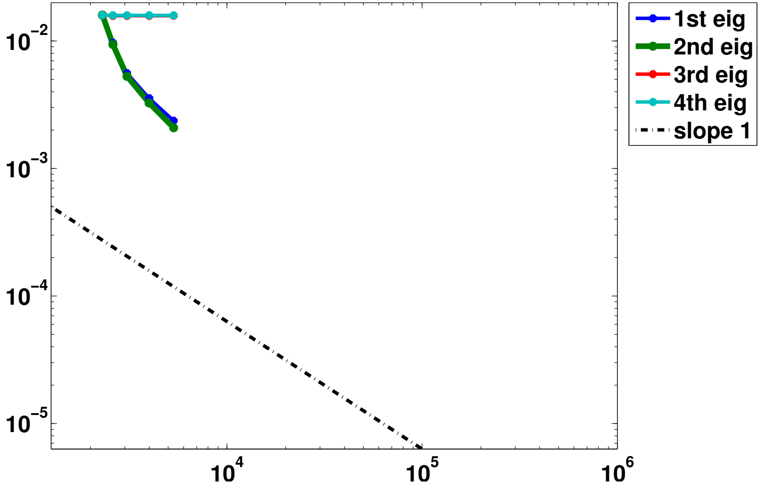
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=3



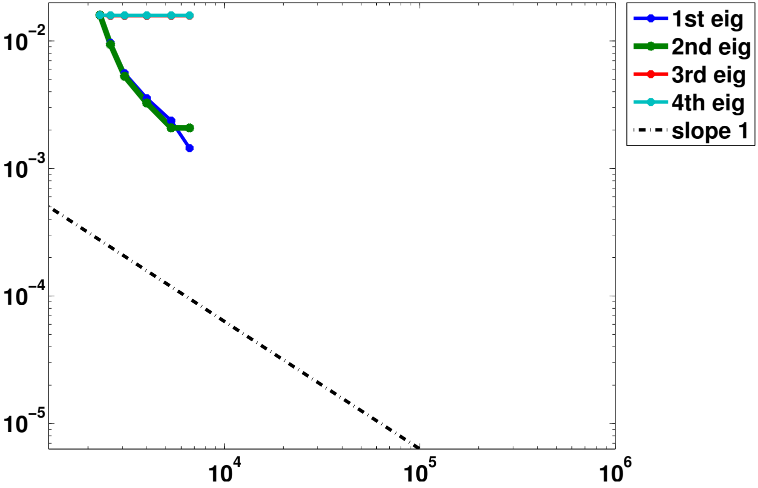
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=4



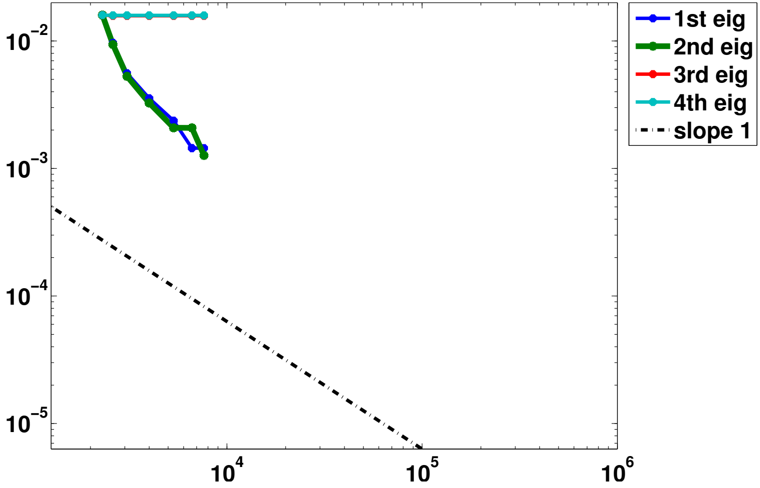
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=5



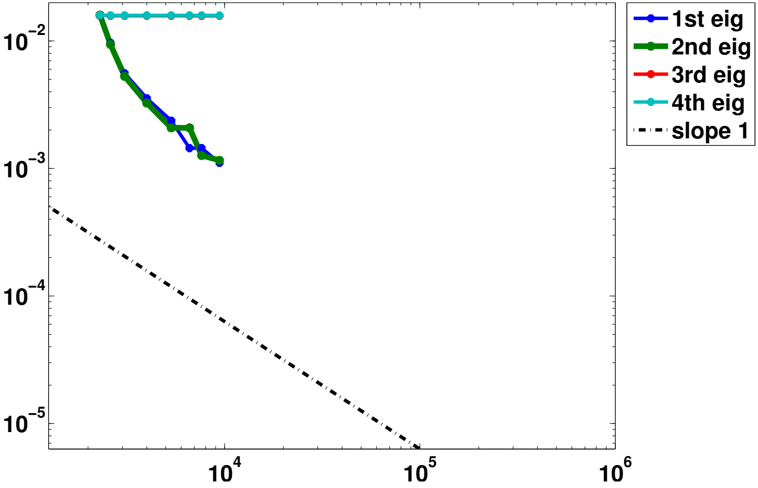
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=6



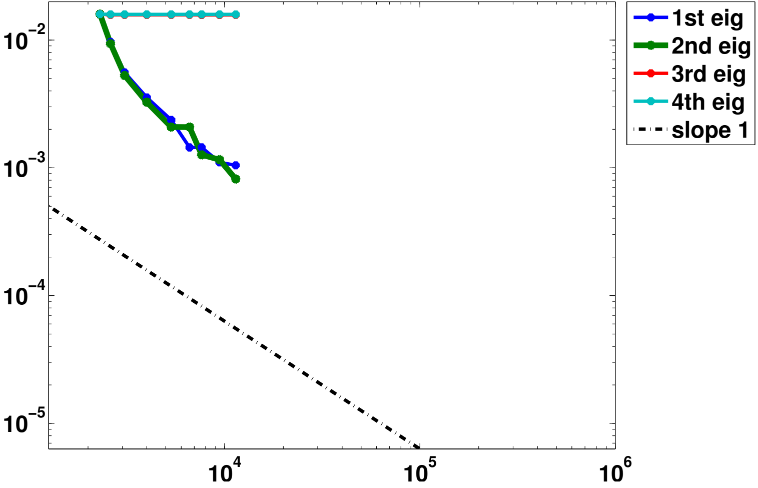
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=7



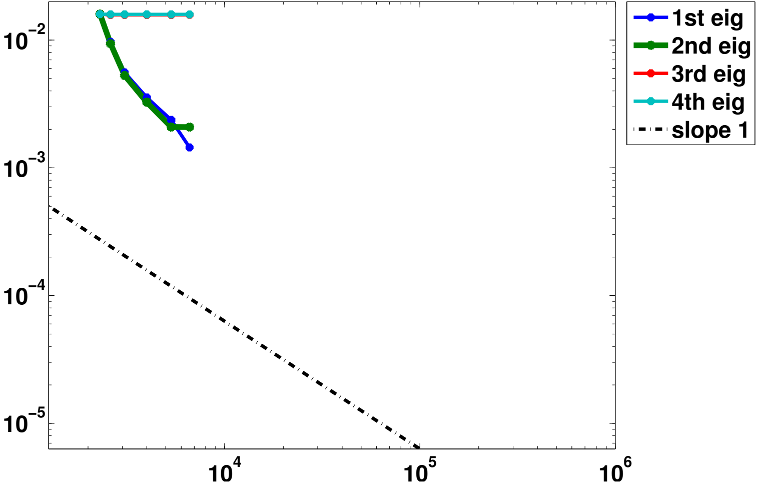
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=8



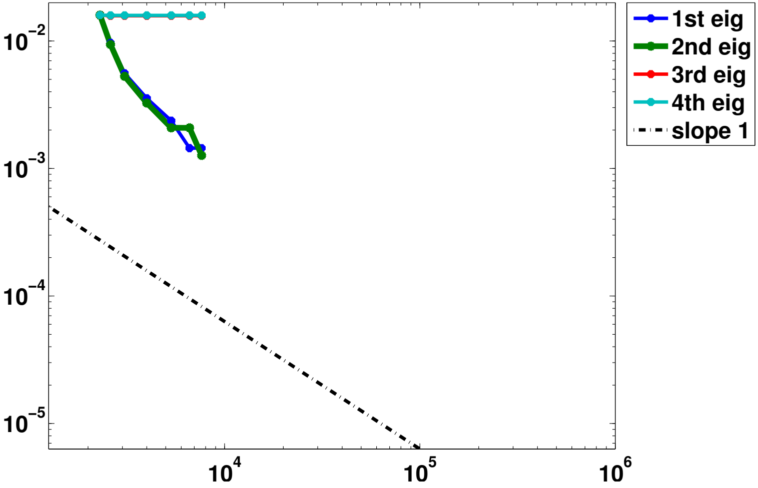
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=5



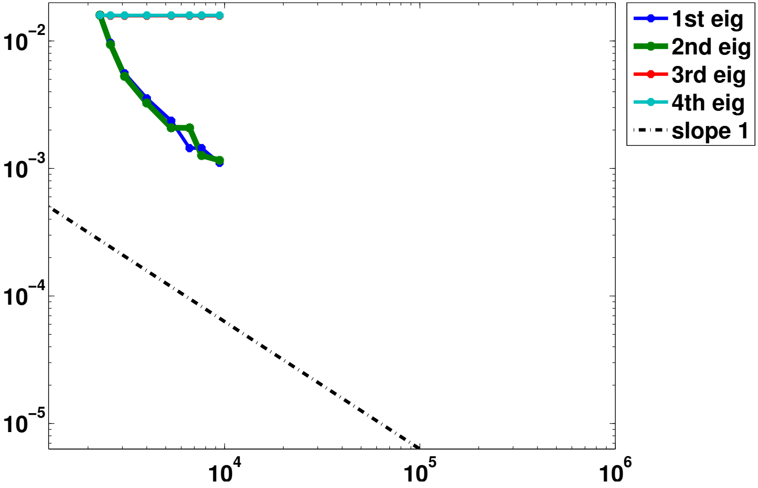
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=6



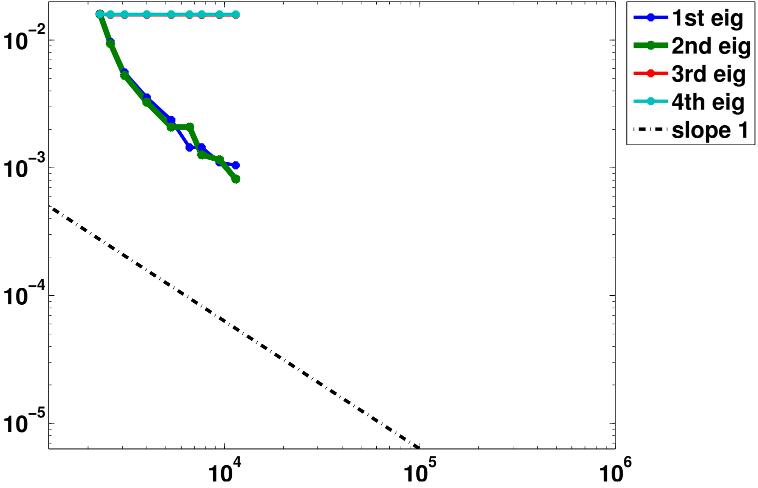
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=7



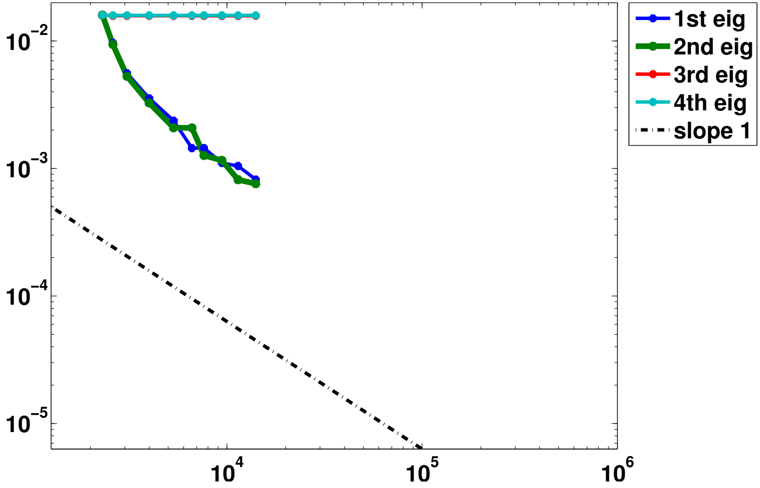
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=8



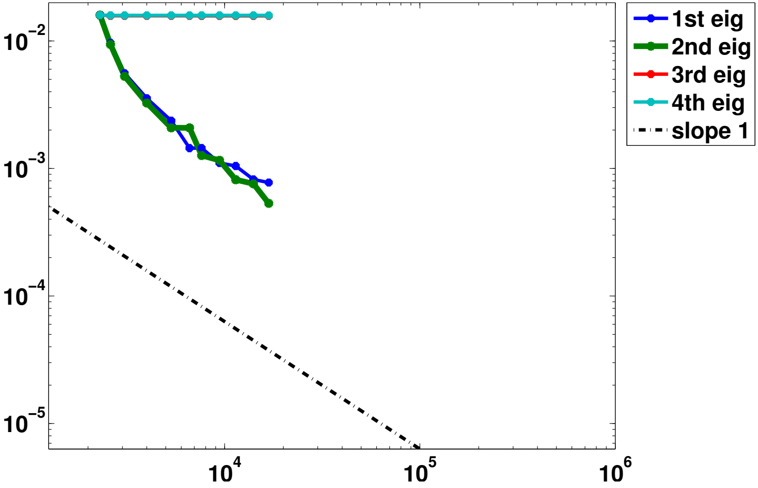
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=9



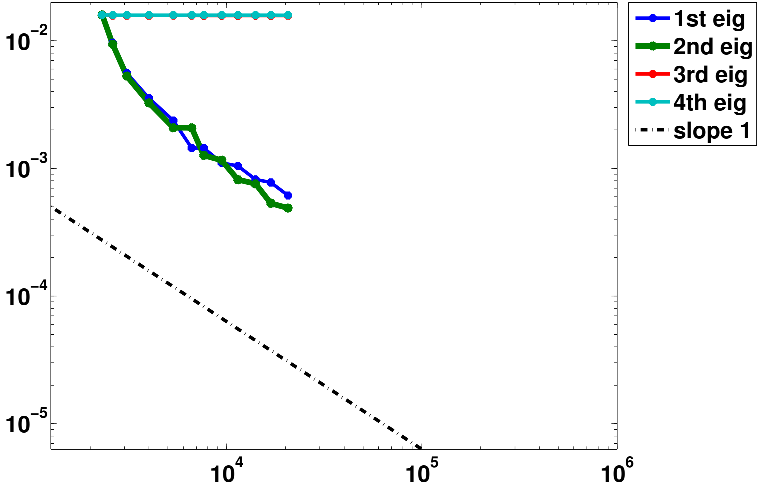
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=10



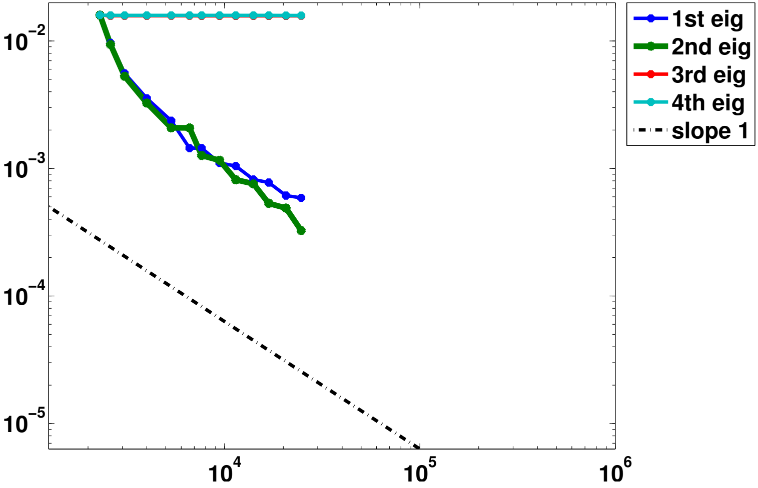
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=11



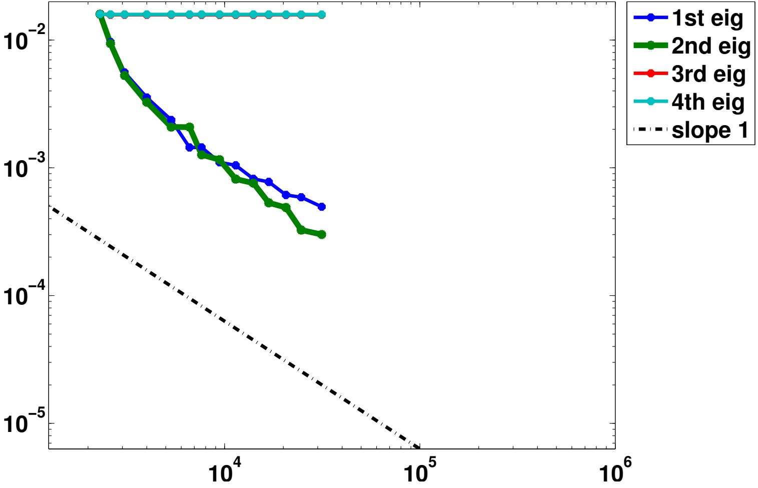
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=12



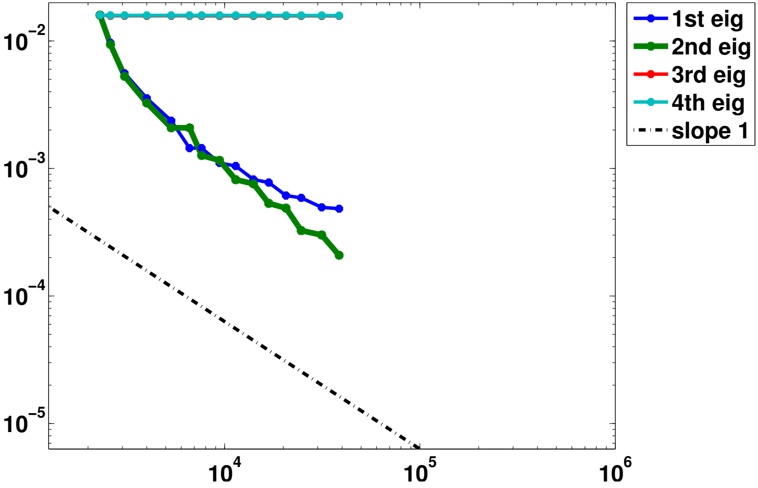
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=13



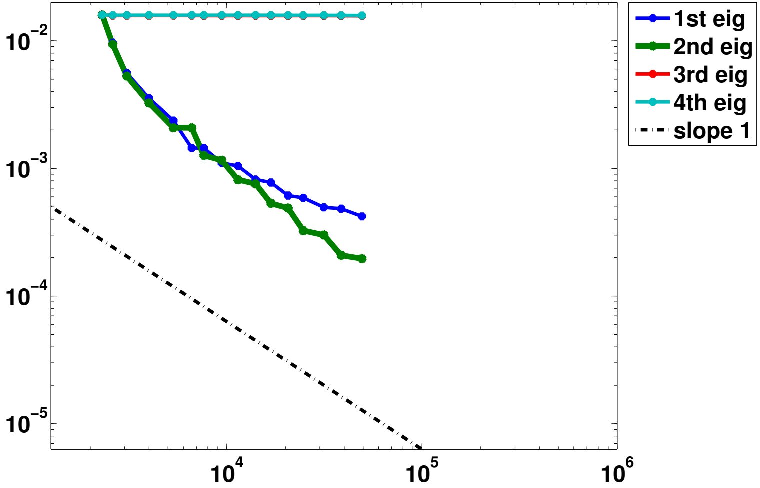
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=14



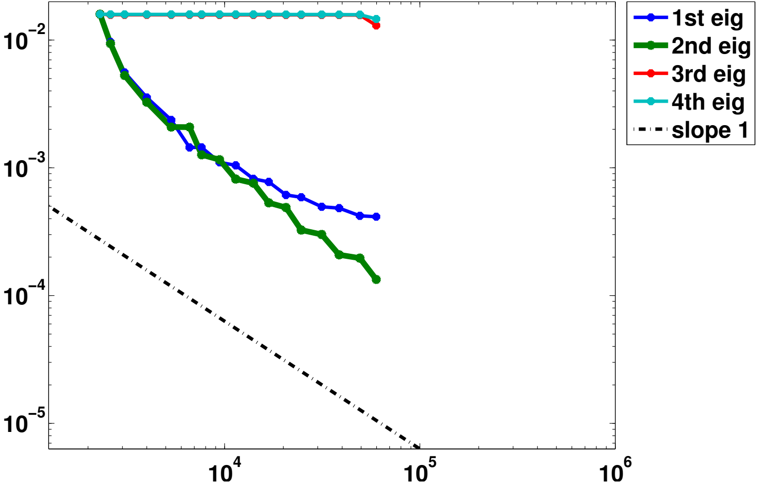
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=15



Approximation of the second frequency

Bulk parameter=0.3, Refinement level=16



Approximation of the second frequency

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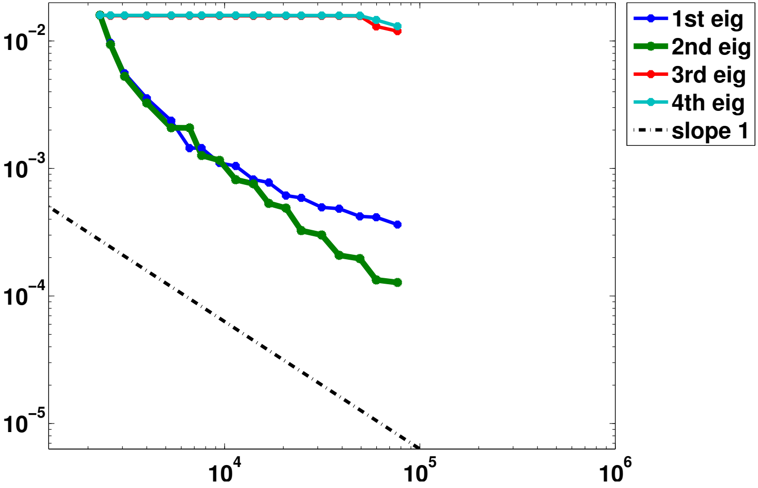
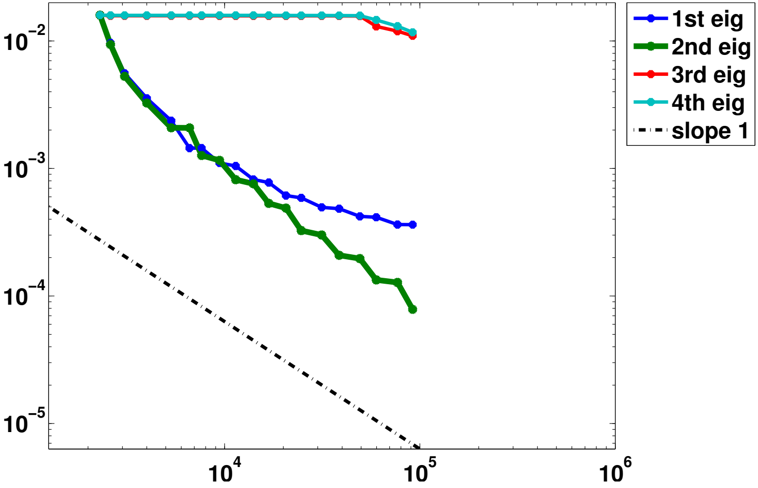


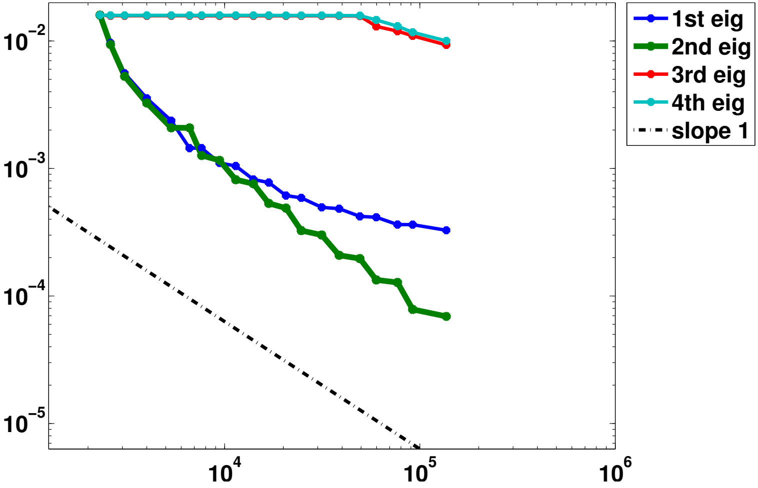
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=18



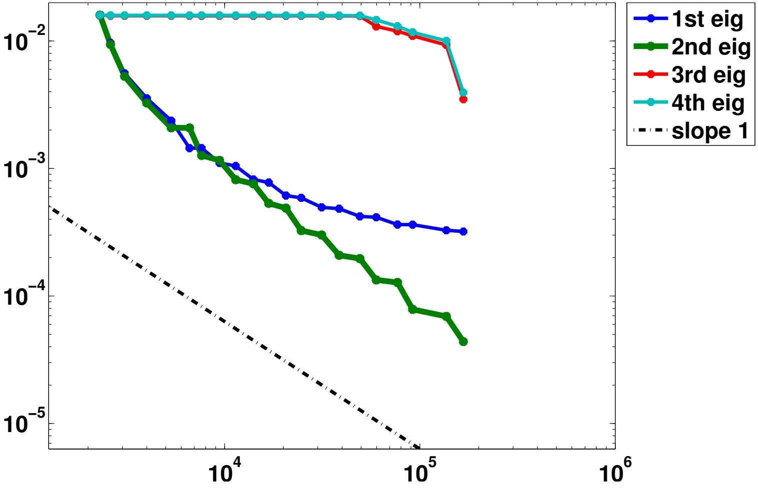
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=19



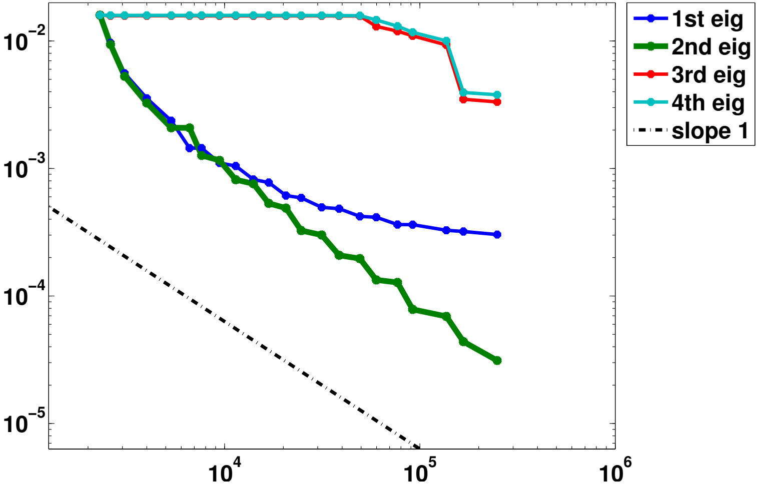
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=20



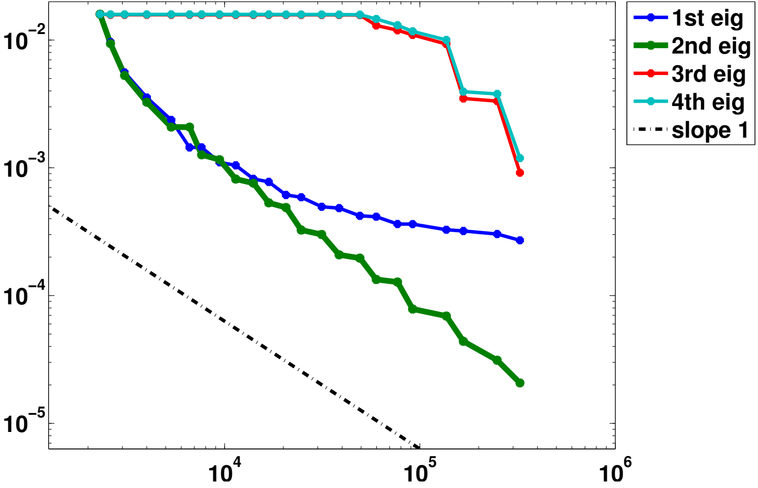
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=21



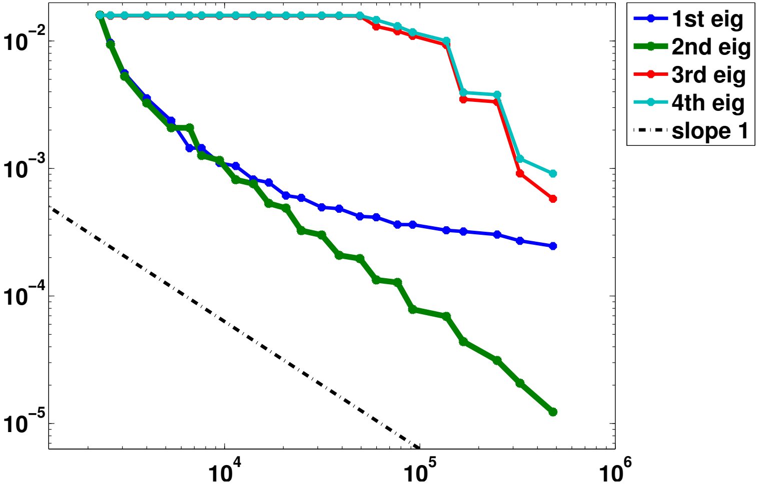
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=22



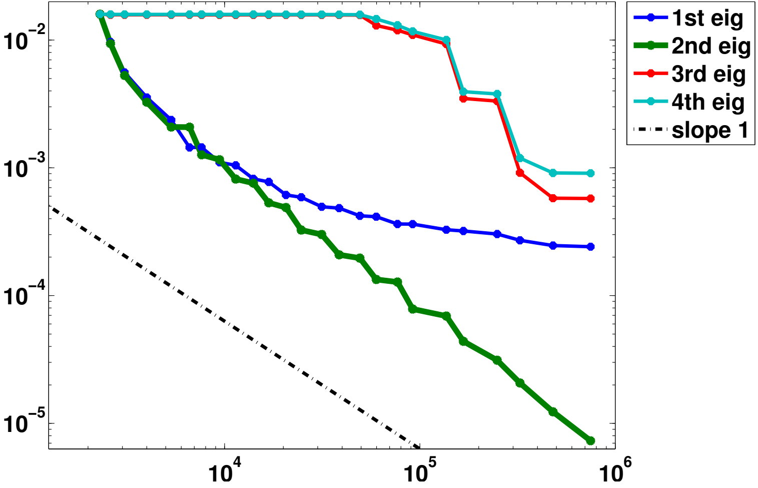
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=23



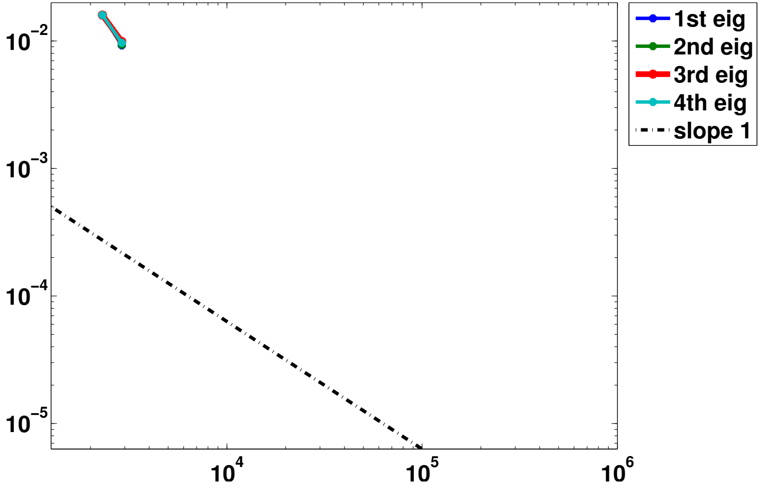
Approximation of the second frequency

Bulk parameter=0.3, Refinement level=24



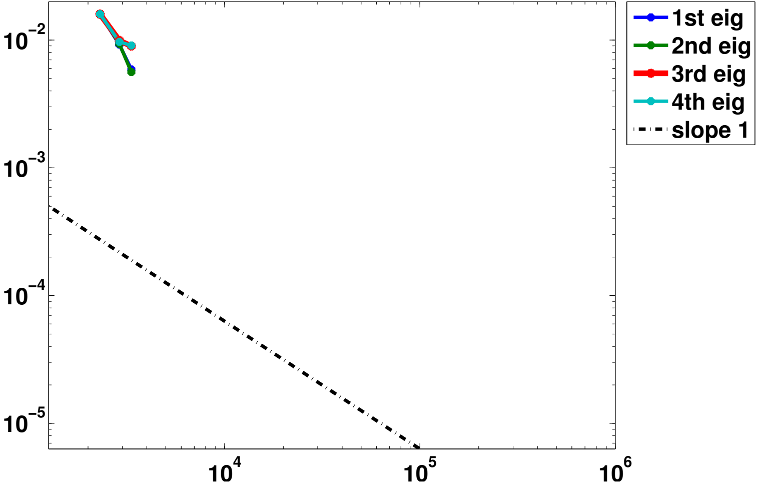
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=1



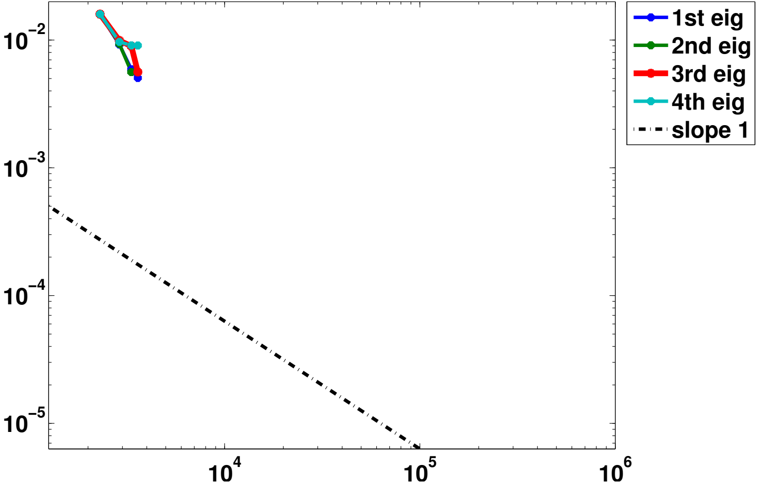
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=2



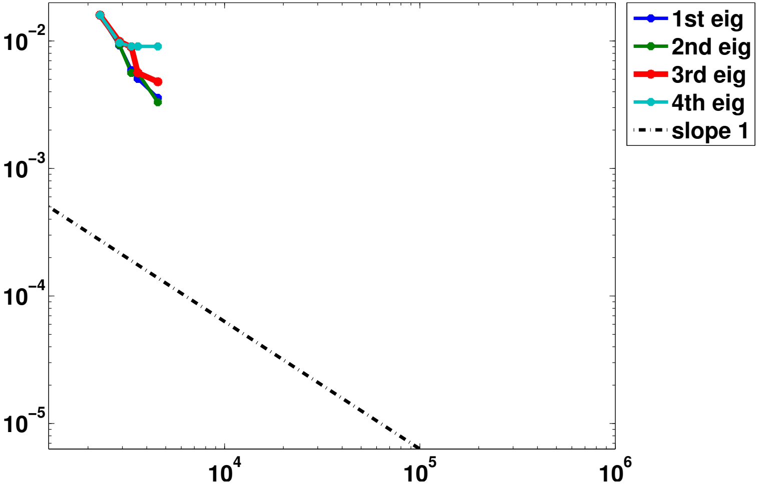
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=3



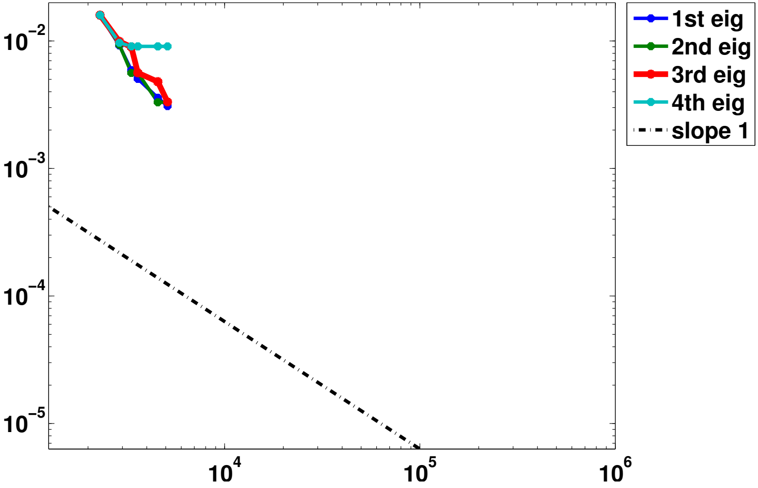
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=4



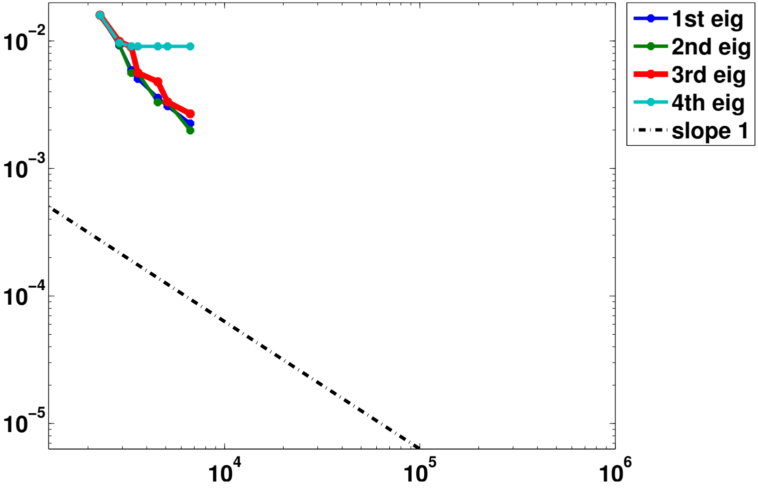
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=5



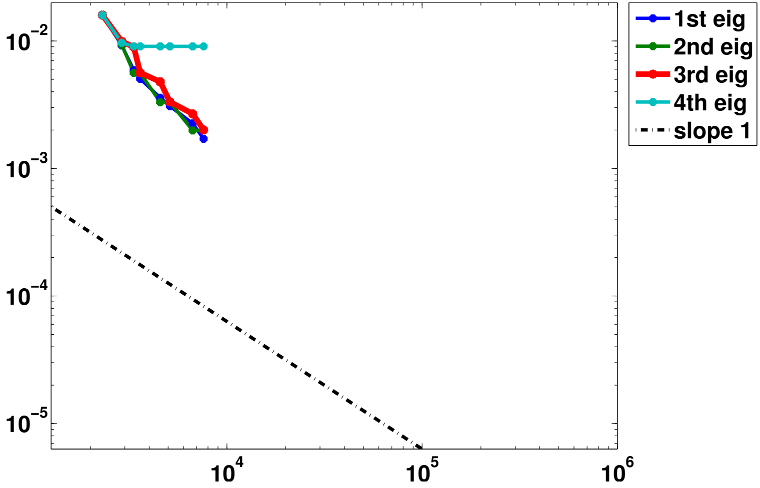
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=6



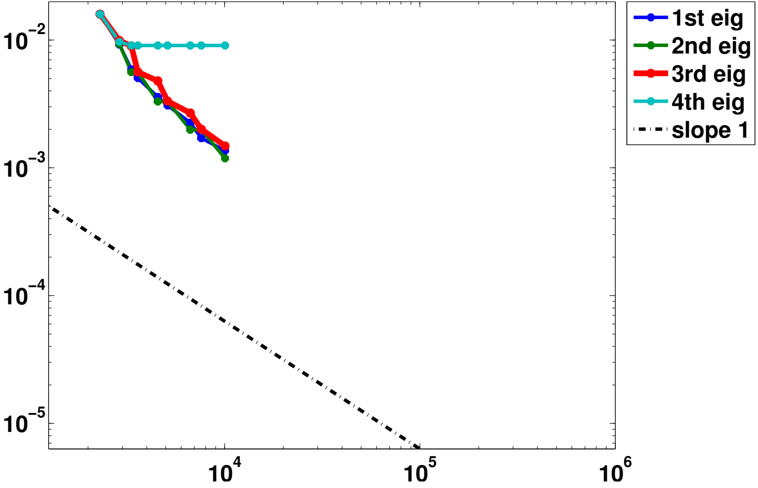
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=7



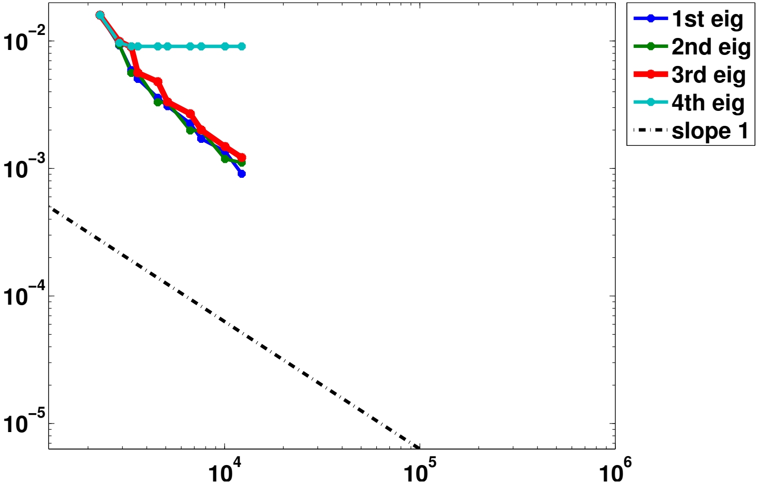
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=8



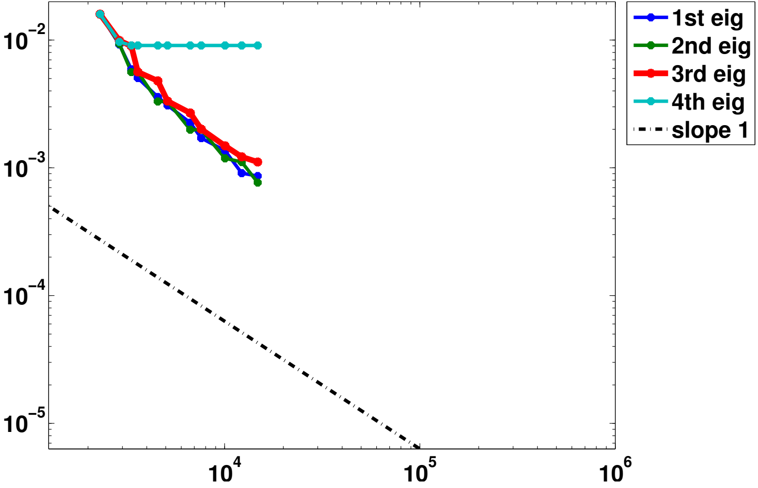
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=9



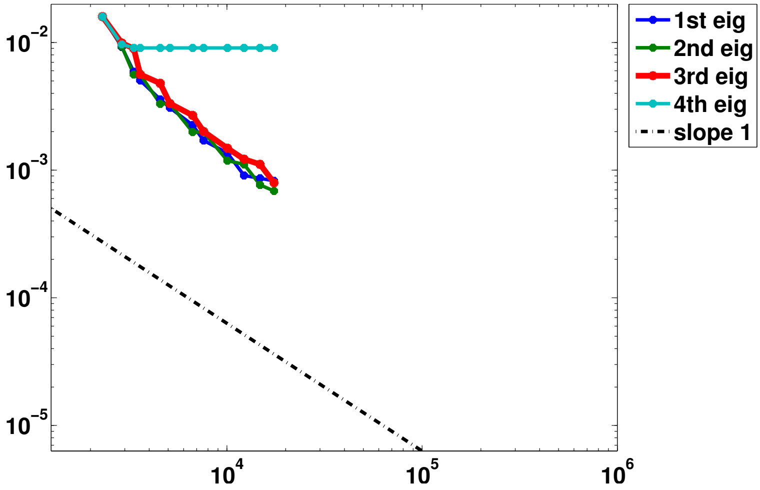
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=10



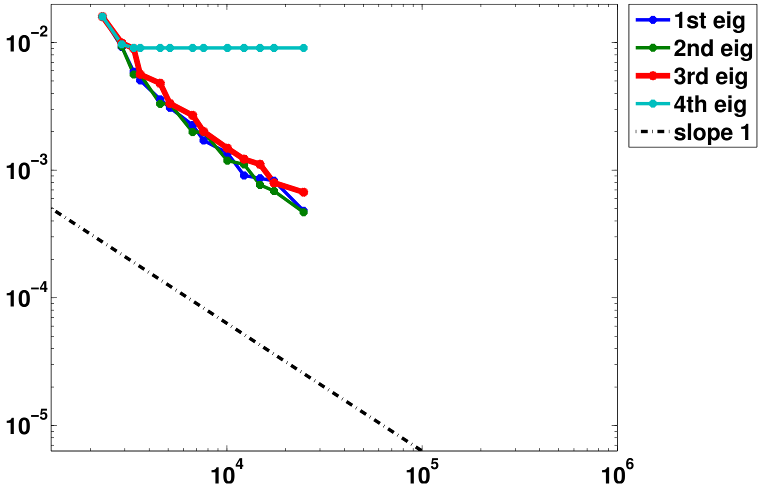
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=11



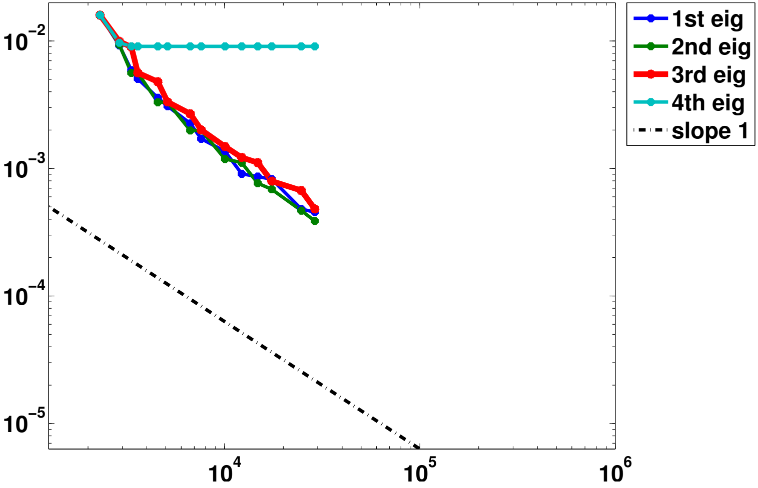
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=12



Approximation of the third frequency

Bulk parameter=0.3, Refinement level=13



Approximation of the third frequency

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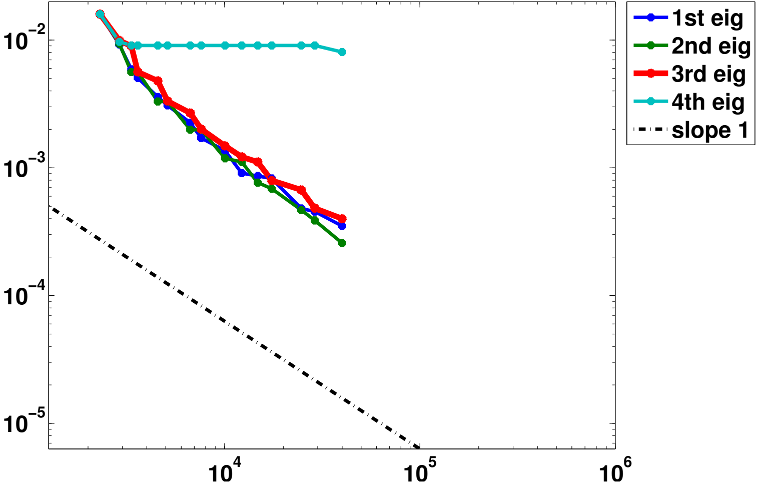
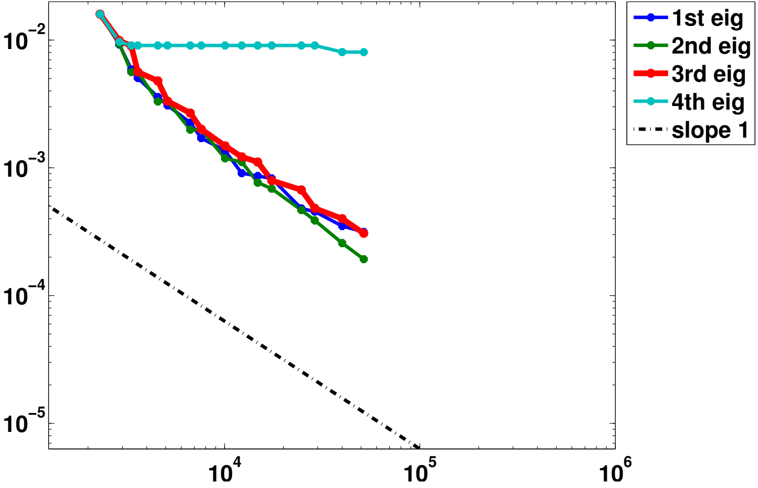


Approximation of the third frequency

Bulk parameter=0.3, Refinement level=15



Approximation of the third frequency

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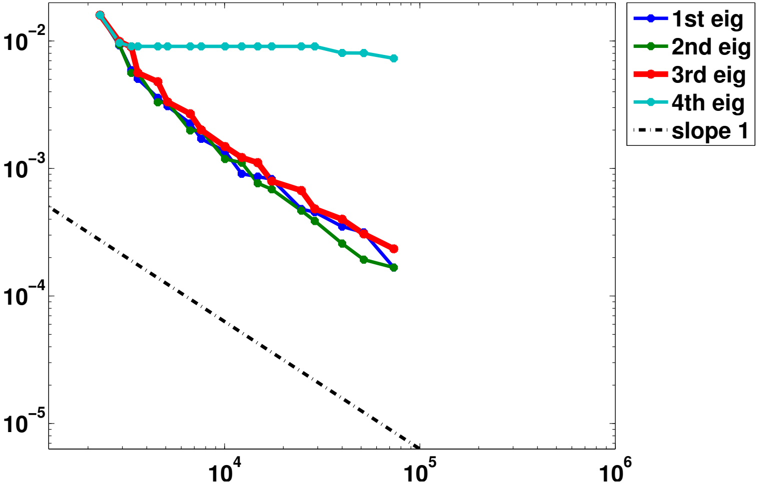

Approximation of the third frequency

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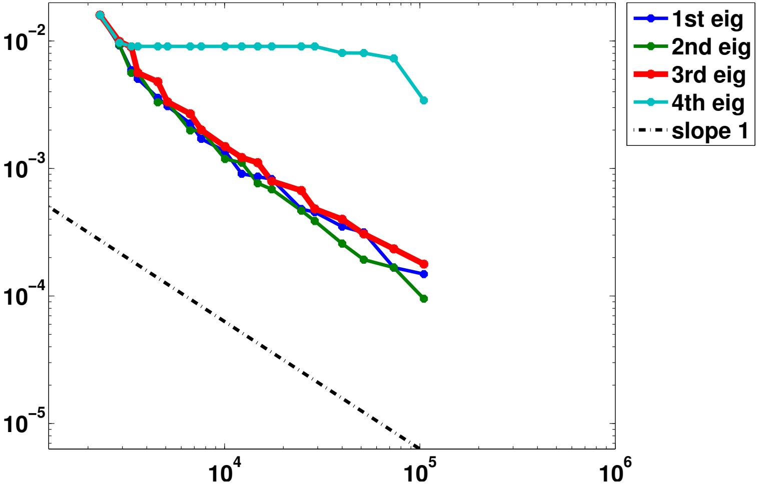
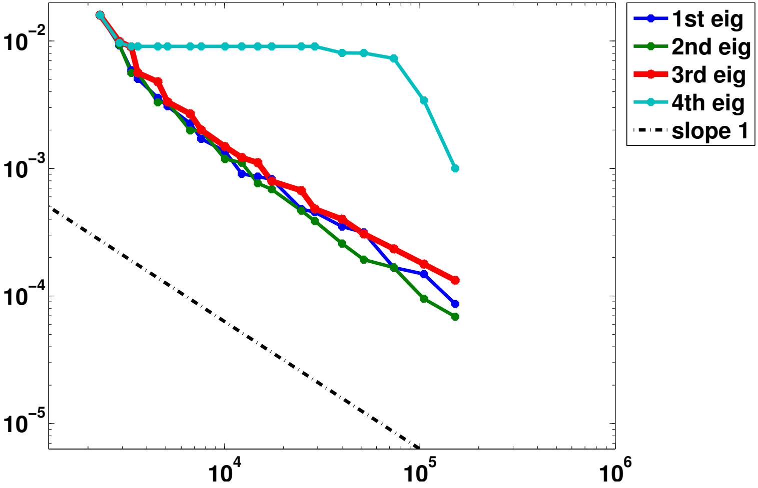


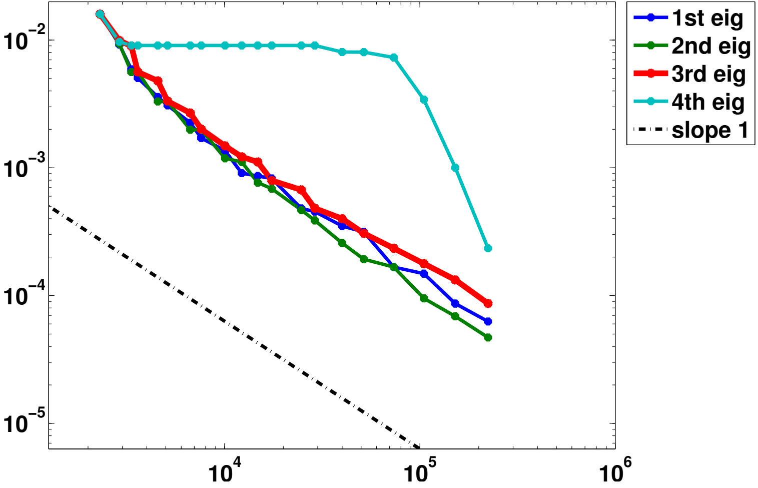
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=18



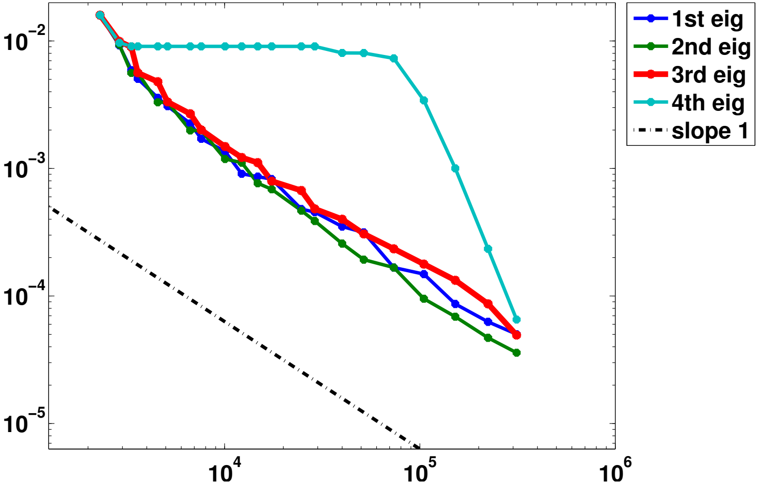
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=19



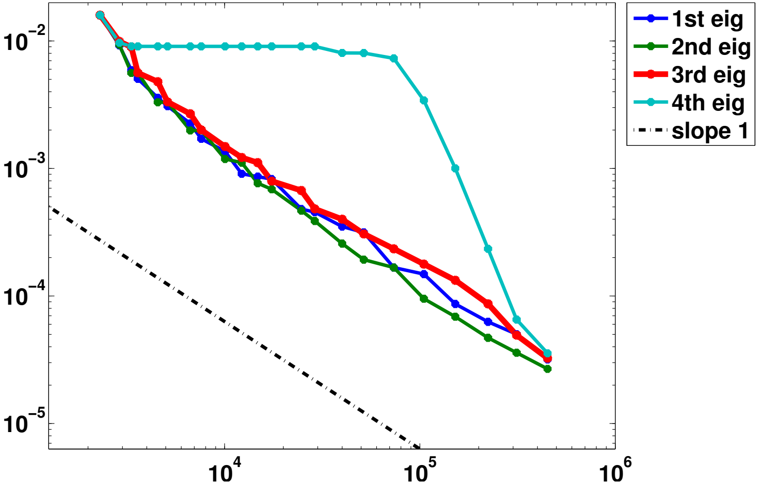
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=20



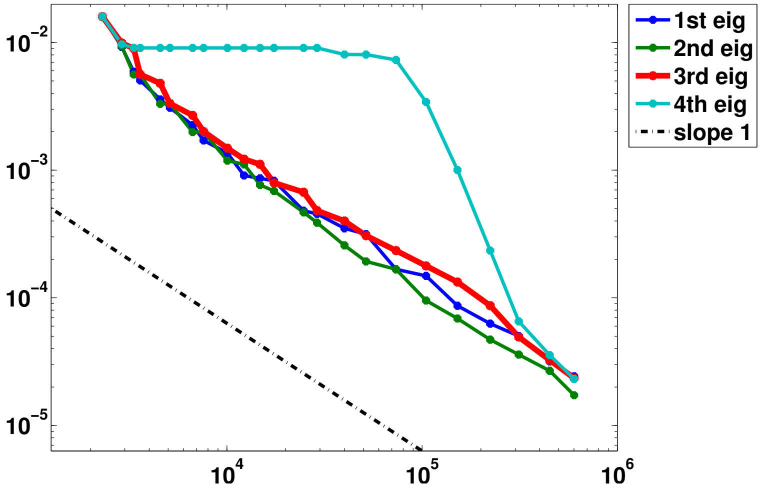
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=21



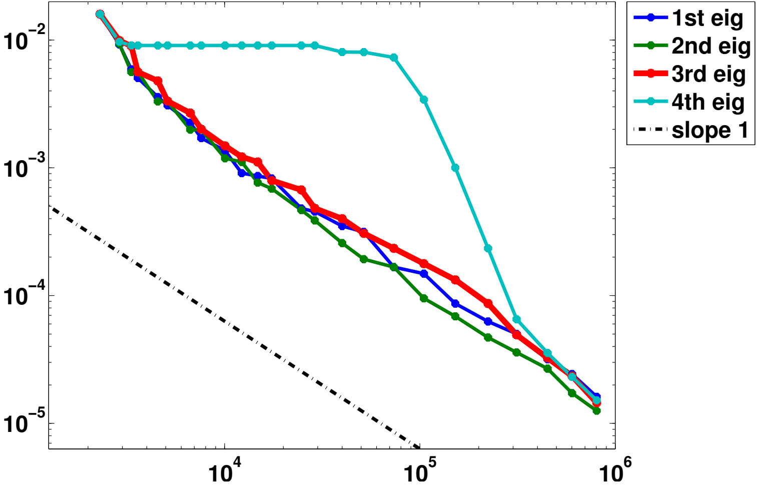
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=22



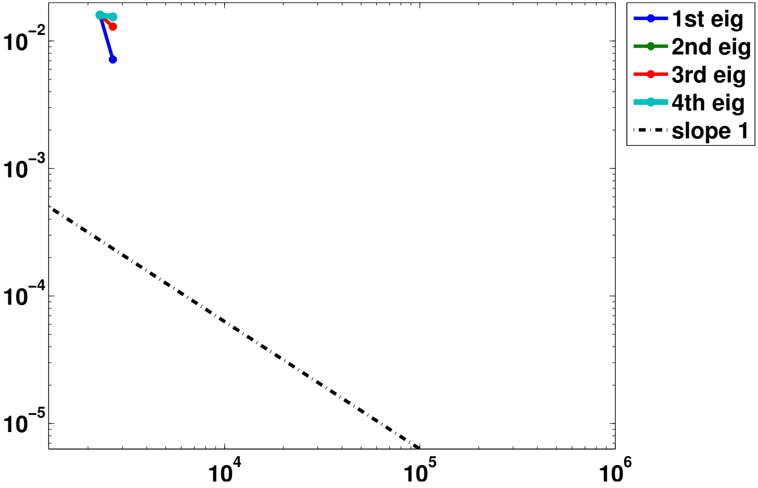
Approximation of the third frequency

Bulk parameter=0.3, Refinement level=23



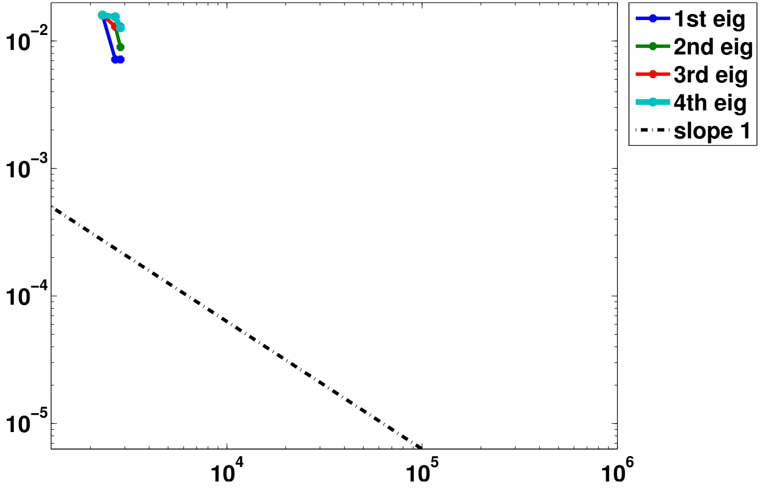
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=1



Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=2



Approximation of the fourth frequency

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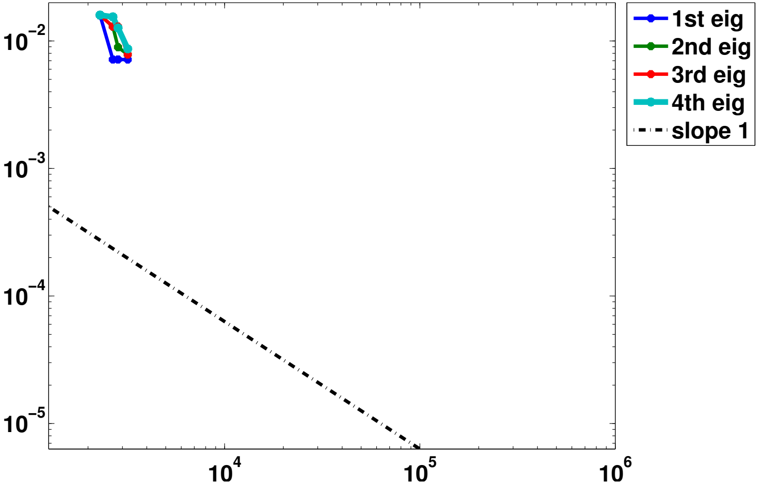
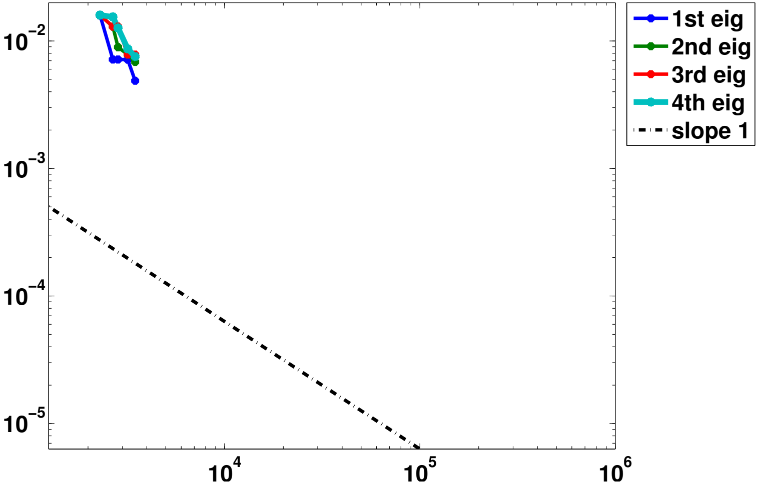


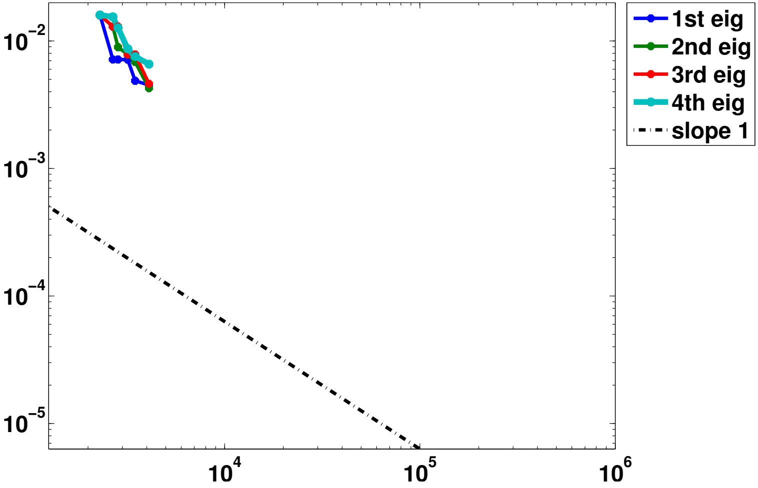
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=4



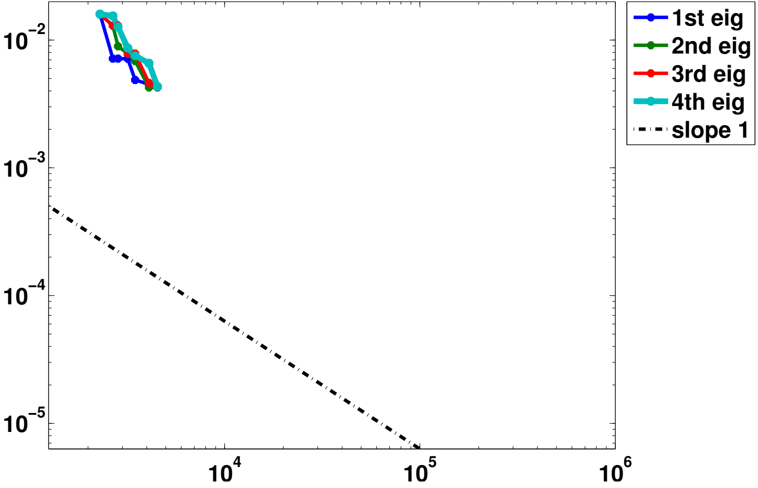
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=5



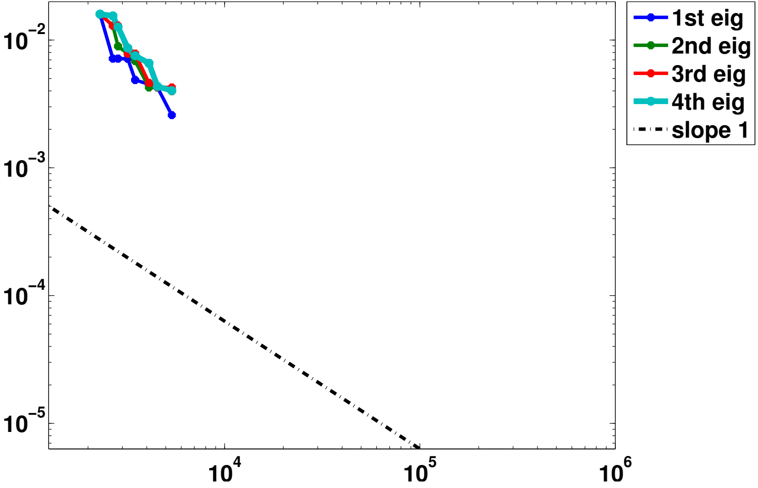
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=6



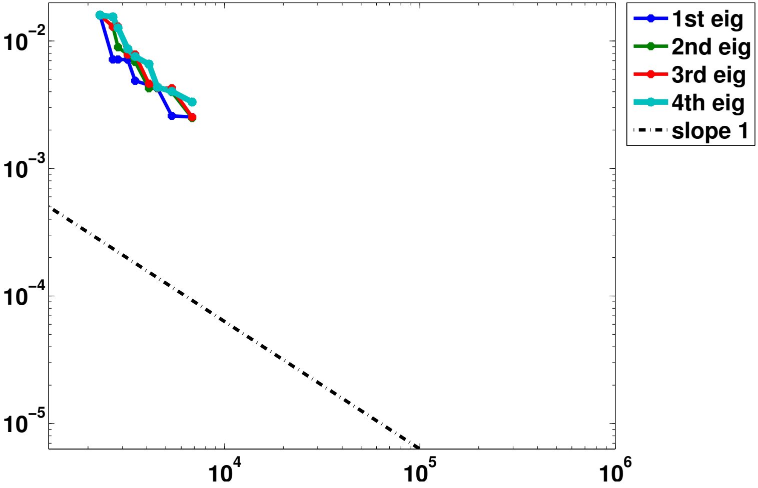
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=7



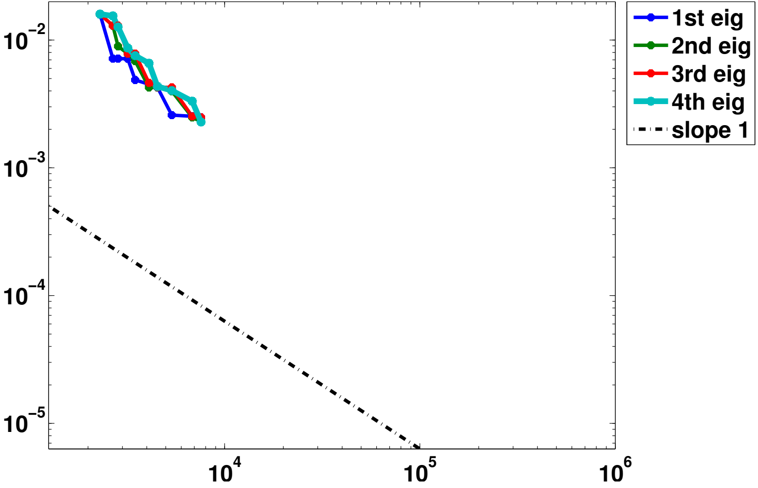
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=8



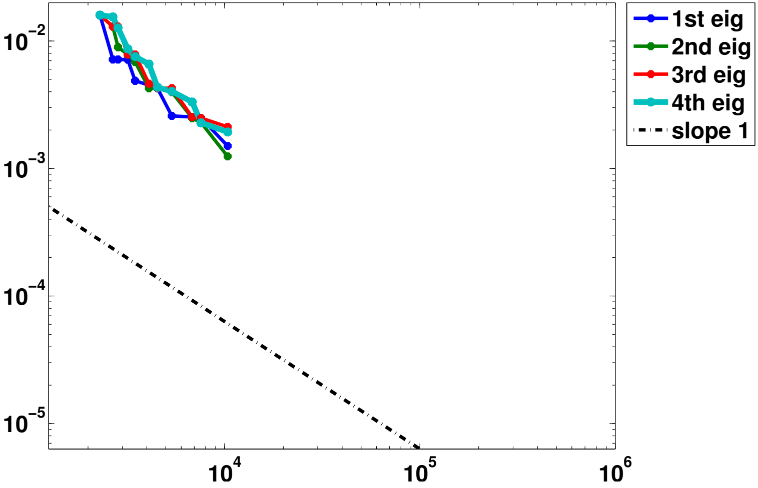
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=9



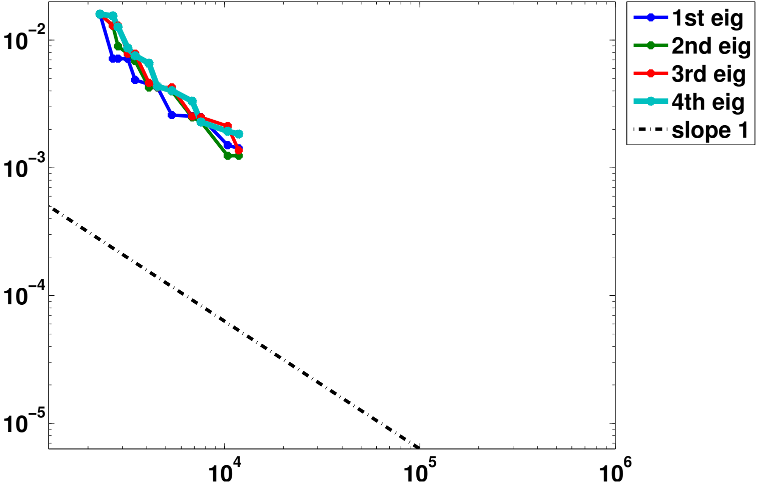
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=10



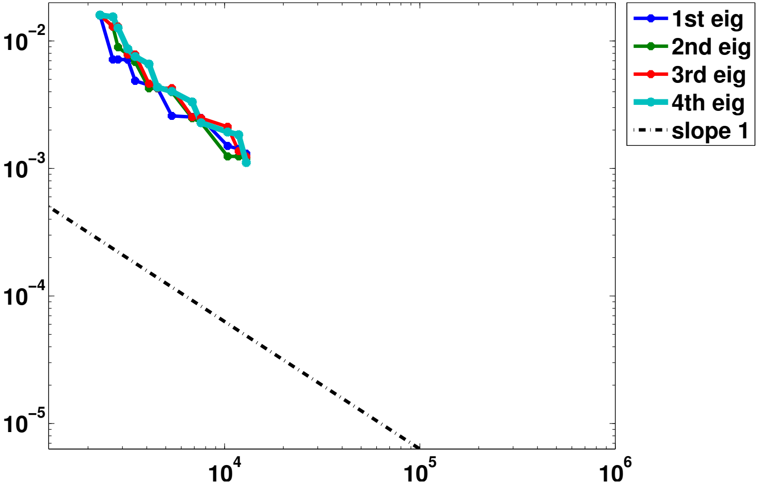
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=11



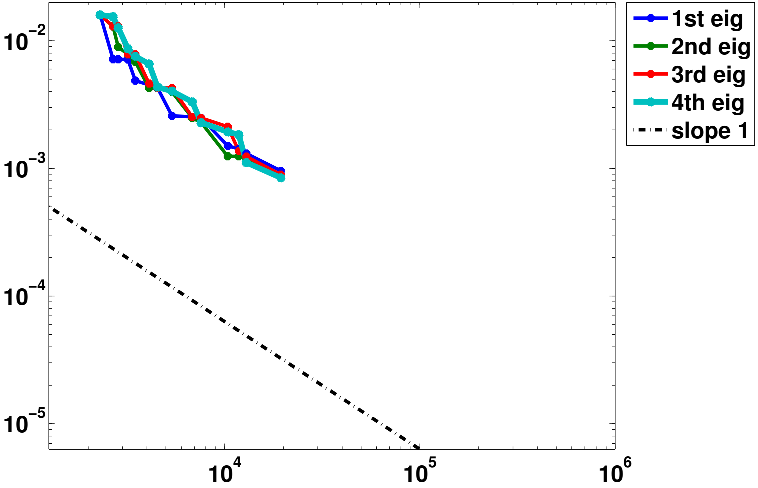
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=12



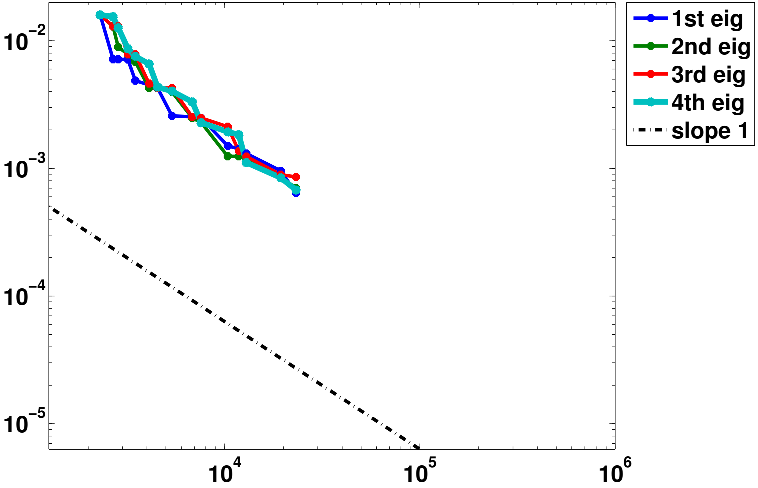
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=13



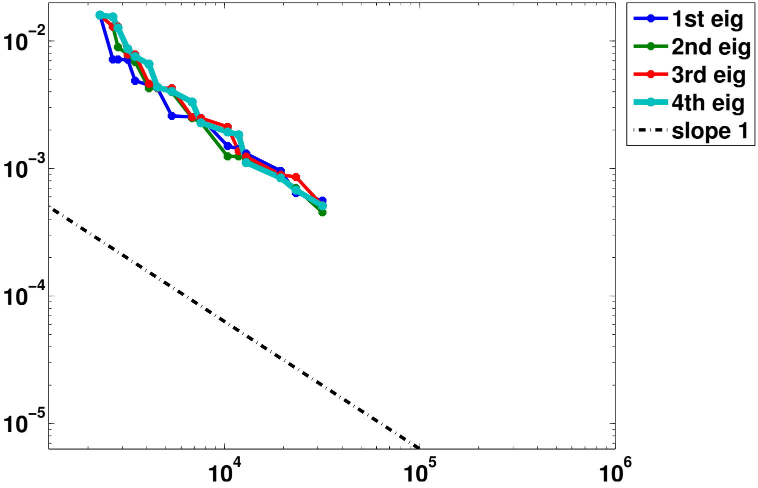
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=14



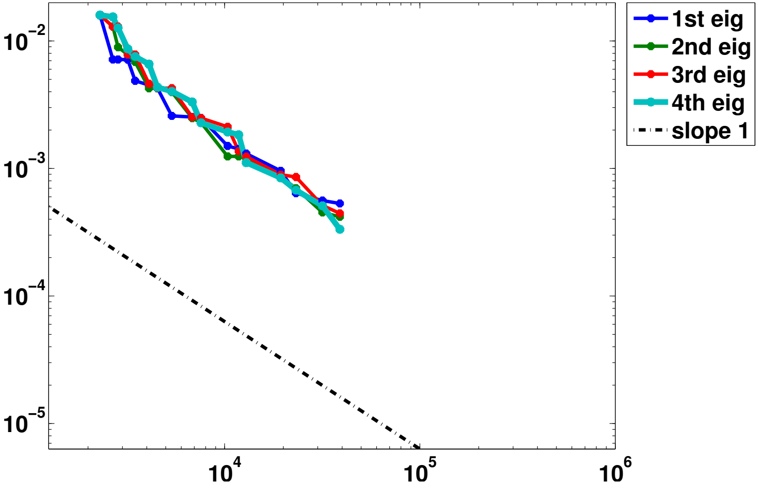
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=15



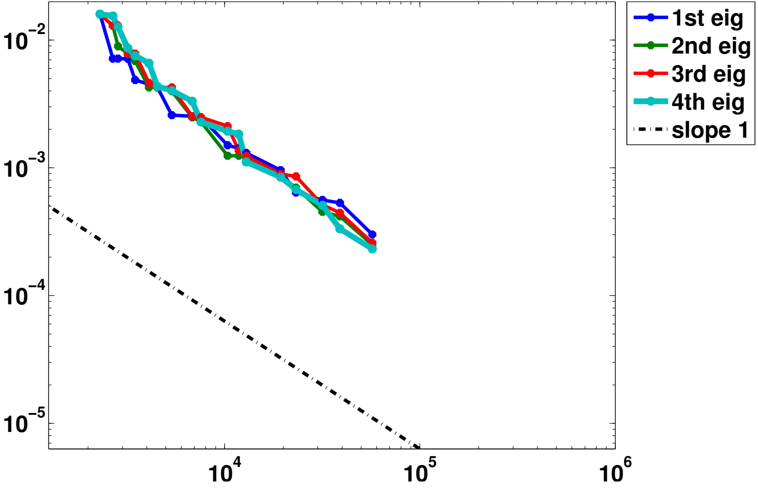
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=16



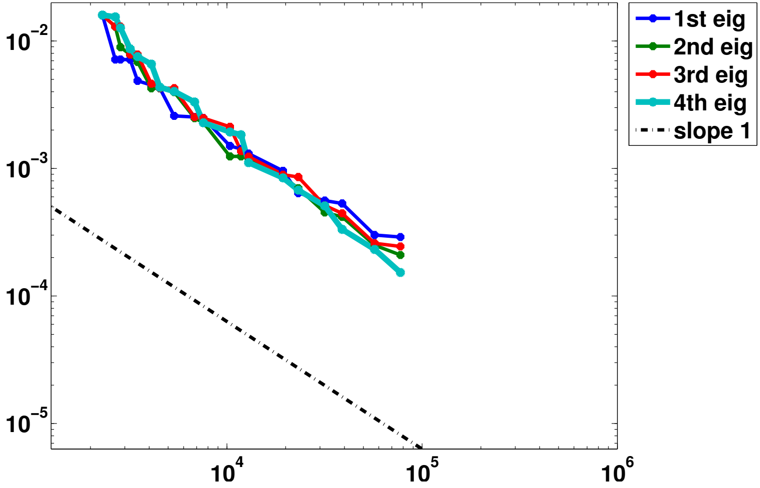
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=17



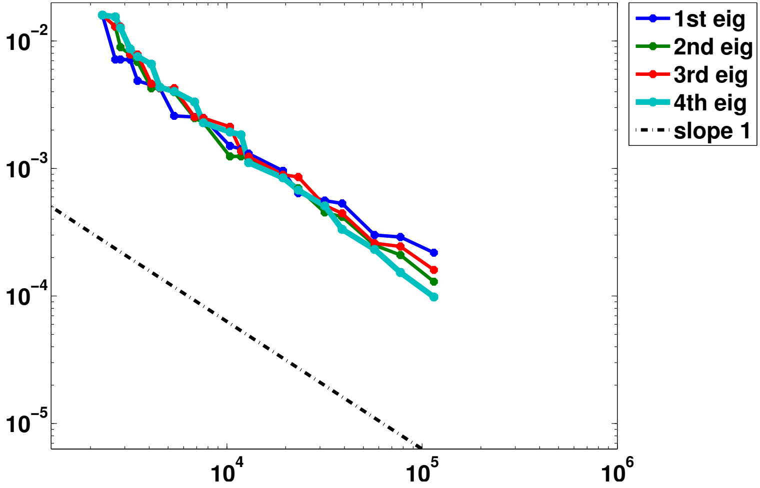
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=18



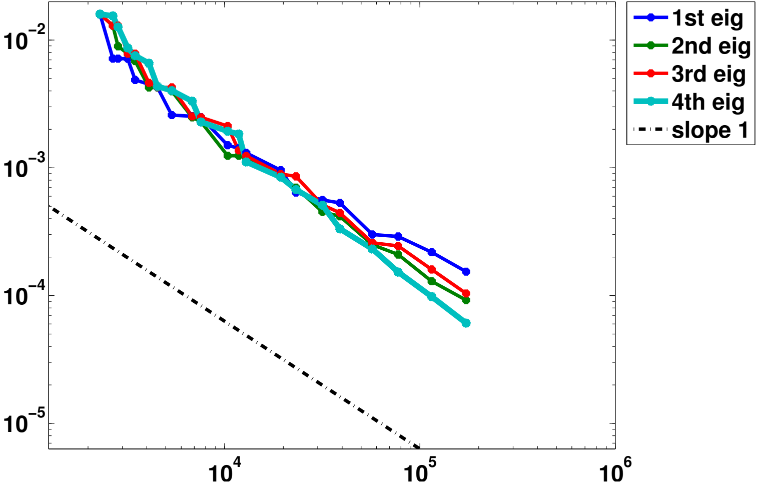
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=19



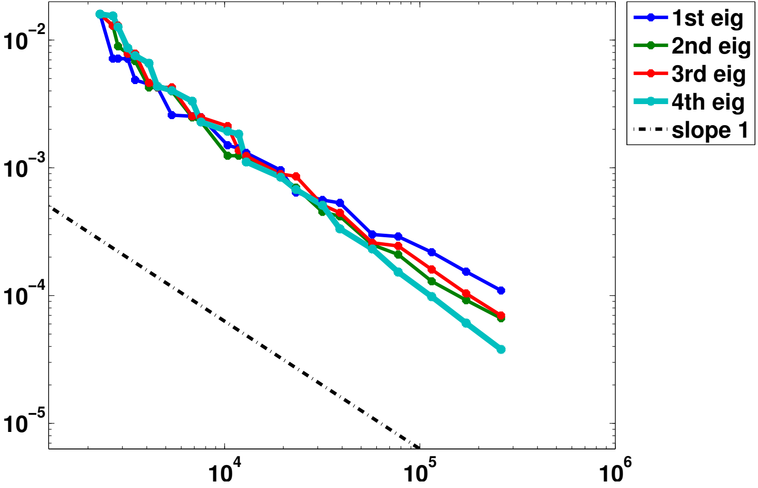
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=20



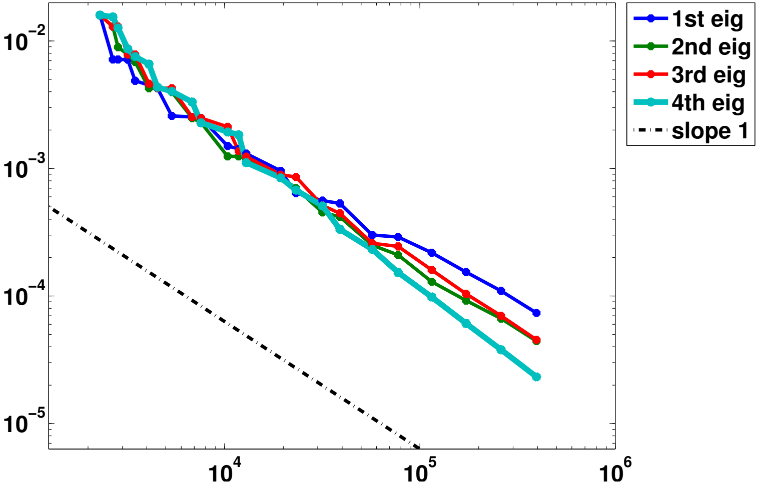
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=21



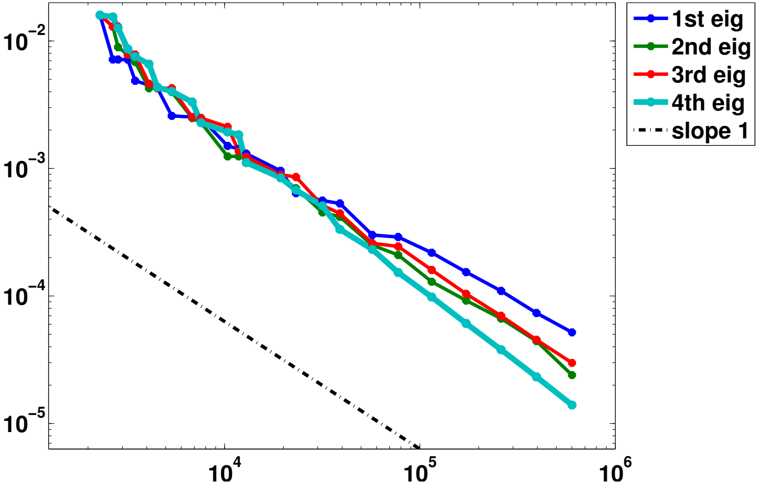
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=22



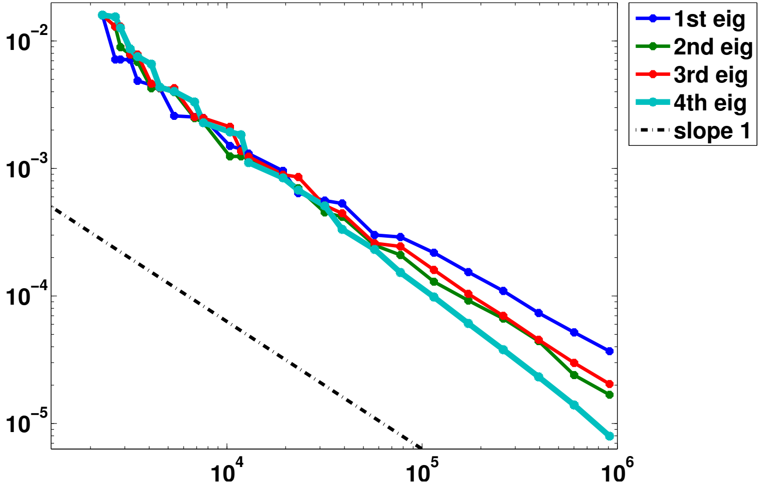
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=23



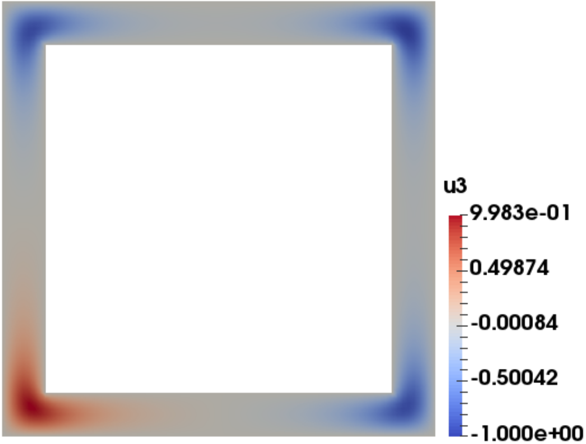
Approximation of the fourth frequency

Bulk parameter=0.3, Refinement level=24



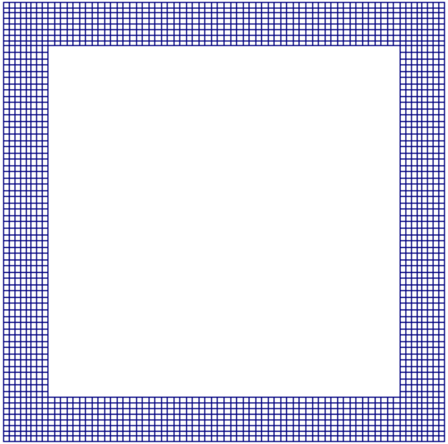
What is going on?

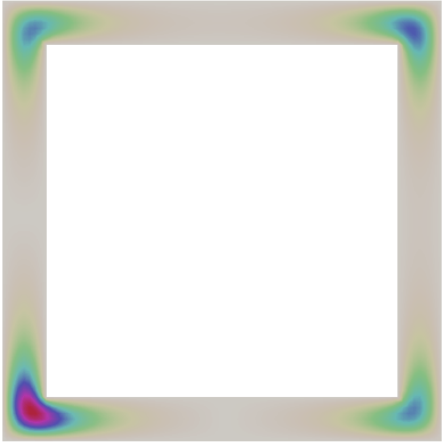
Let's see the mesh sequence and the computed eigenfunctions, for instance, in the case of the third frequency



What is going on?

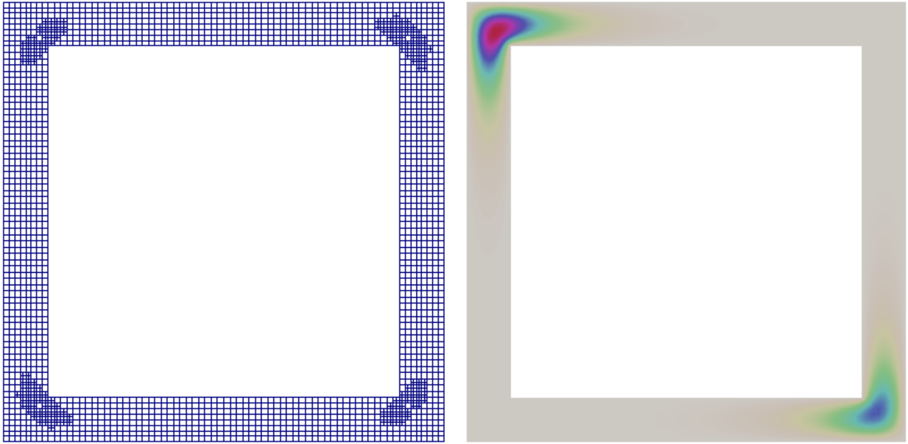
Initial mesh





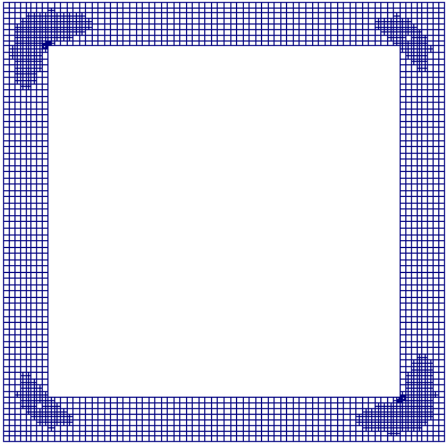
What is going on?

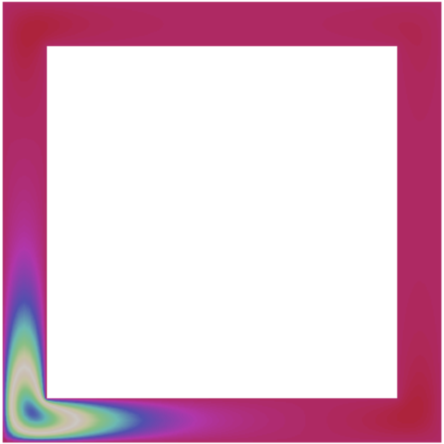
Refinement level=1



What is going on?

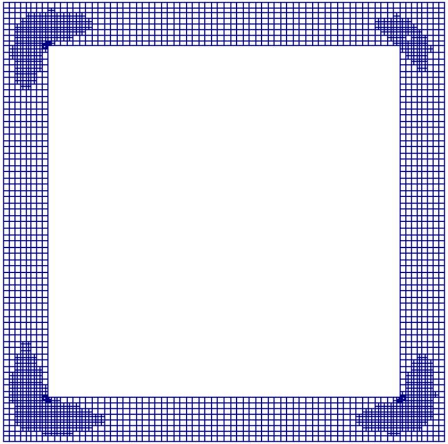
Refinement level=2

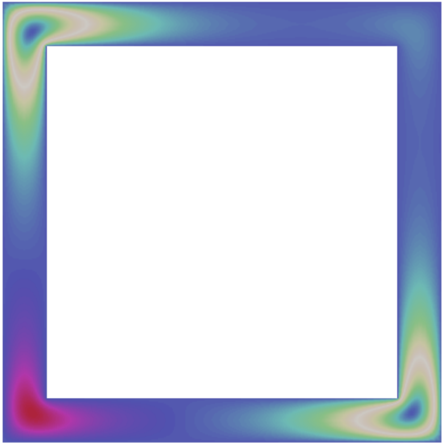




What is going on?

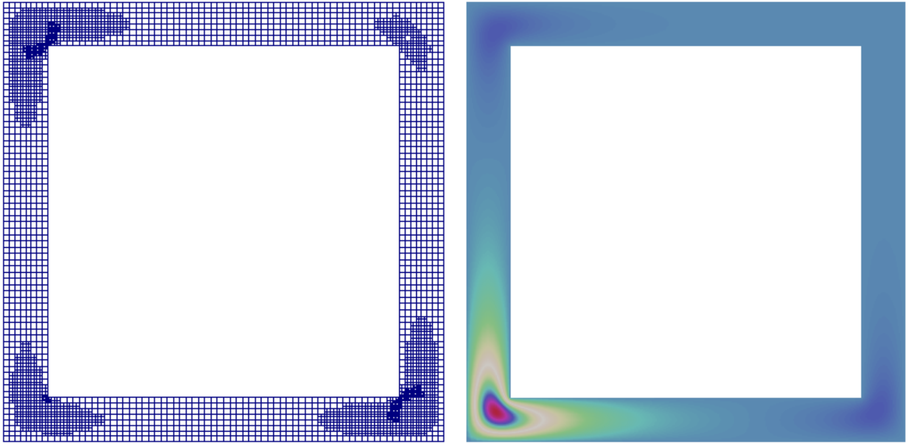
Refinement level=3





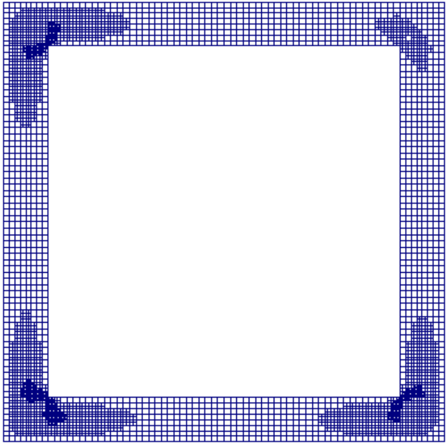
What is going on?

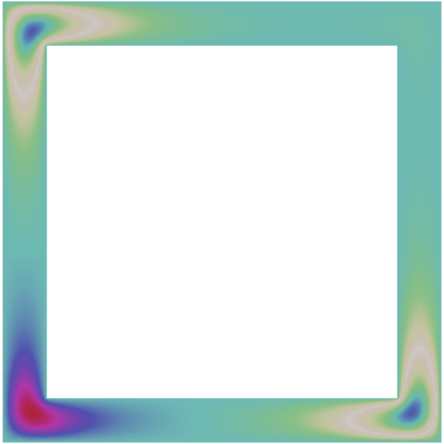
Refinement level=4



What is going on?

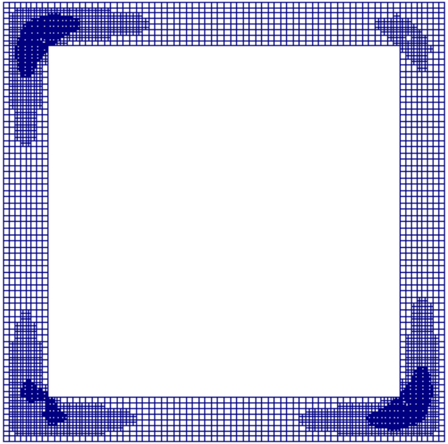
Refinement level=5

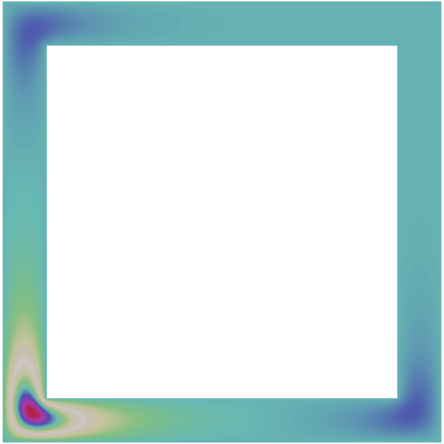




What is going on?

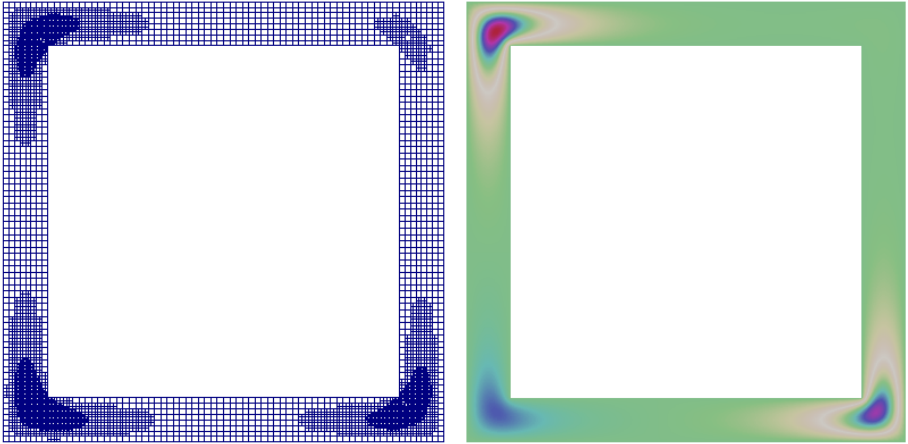
Refinement level=6





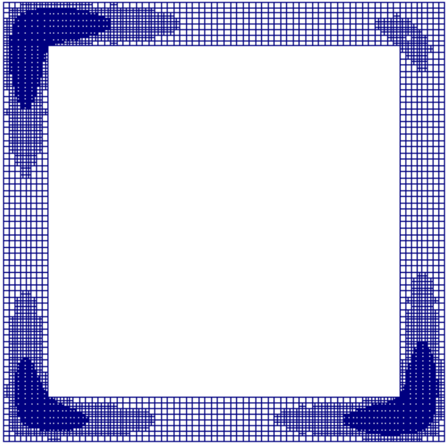
What is going on?

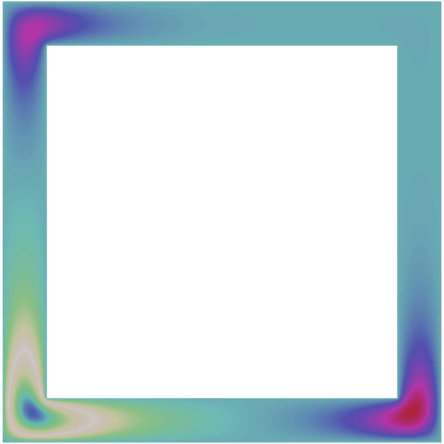
Refinement level=7



What is going on?

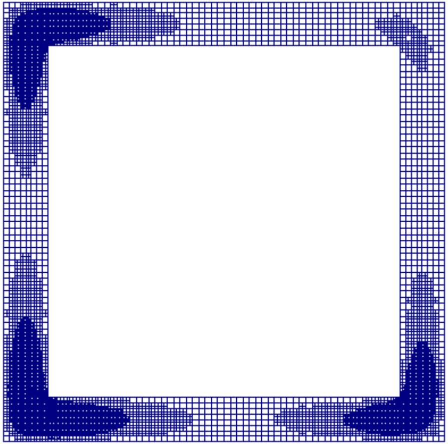
Refinement level=8

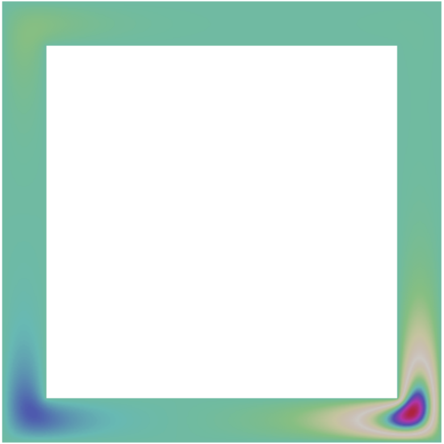




What is going on?

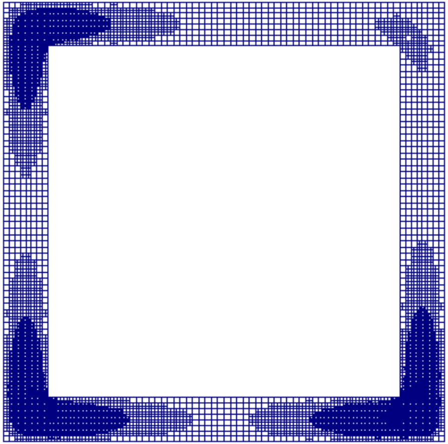
Refinement level=9

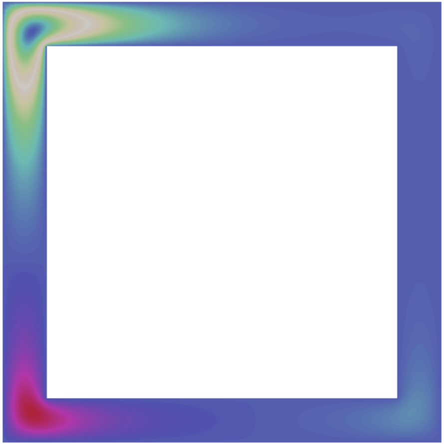




What is going on?

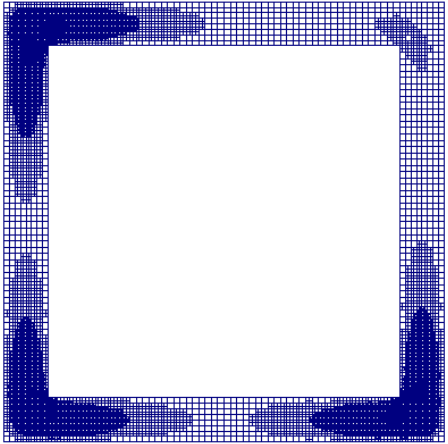
Refinement level=10

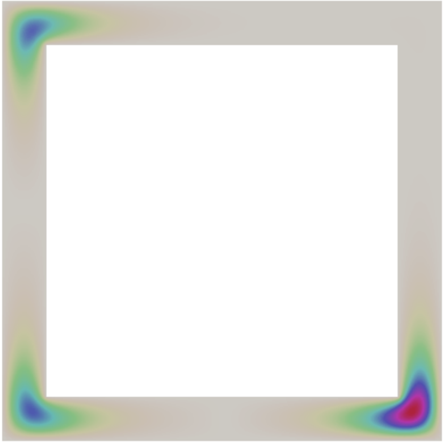




What is going on?

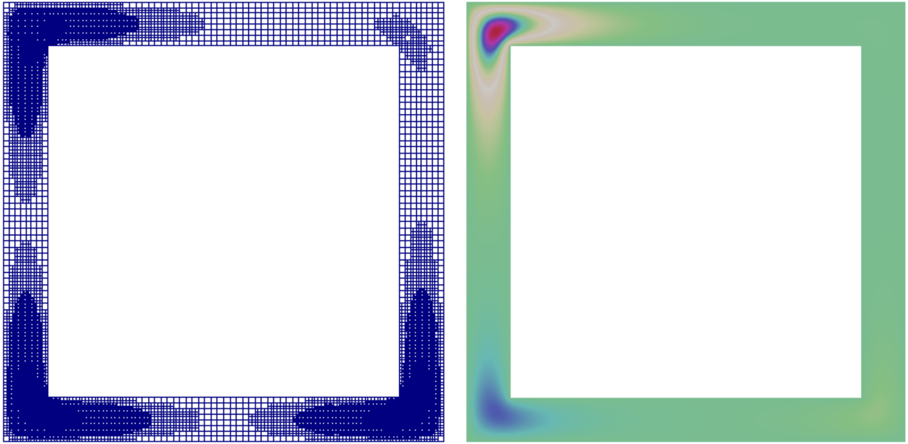
Refinement level=11





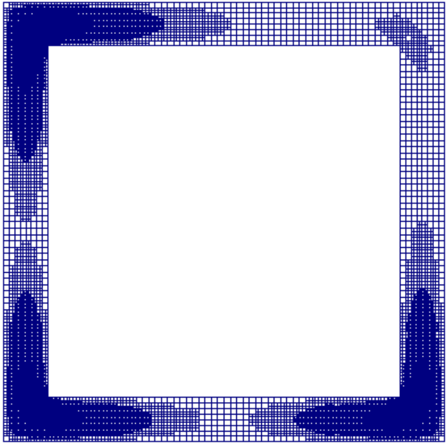
What is going on?

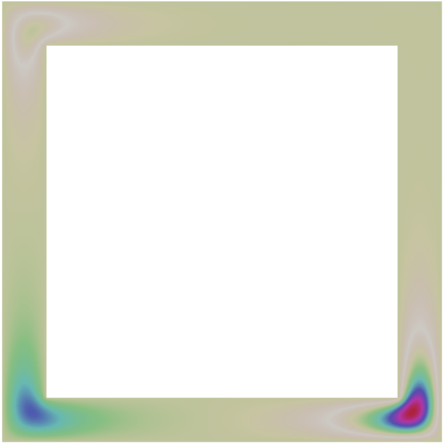
Refinement level=12



What is going on?

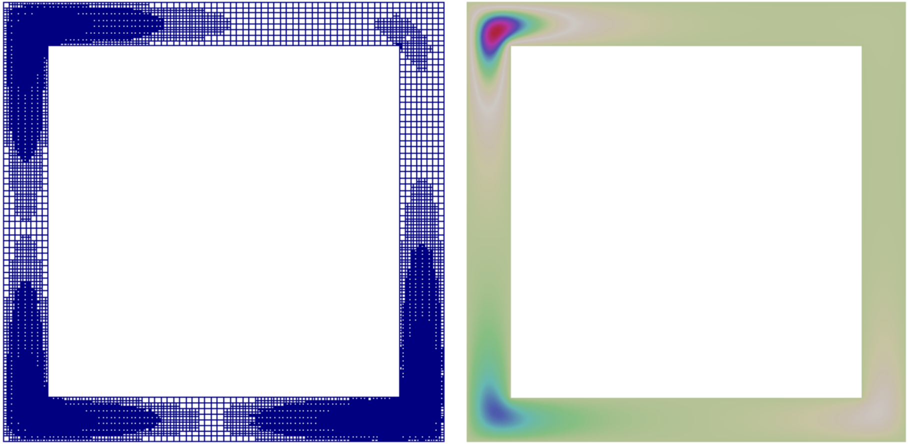
Refinement level=13





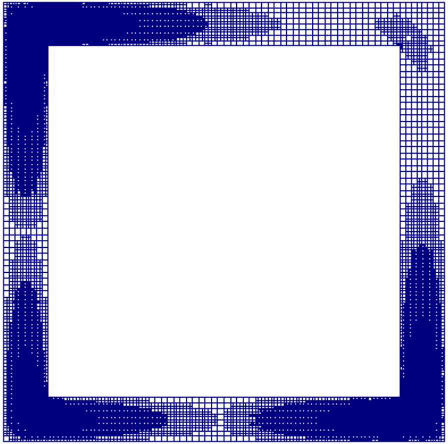
What is going on?

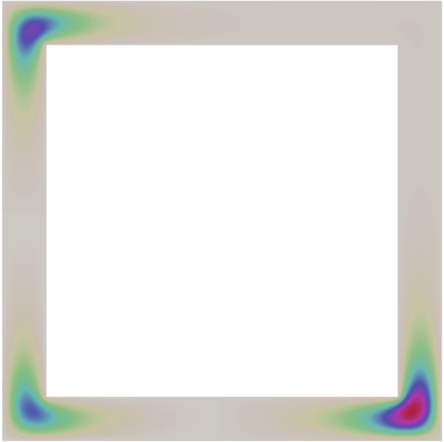
Refinement level=14



What is going on?

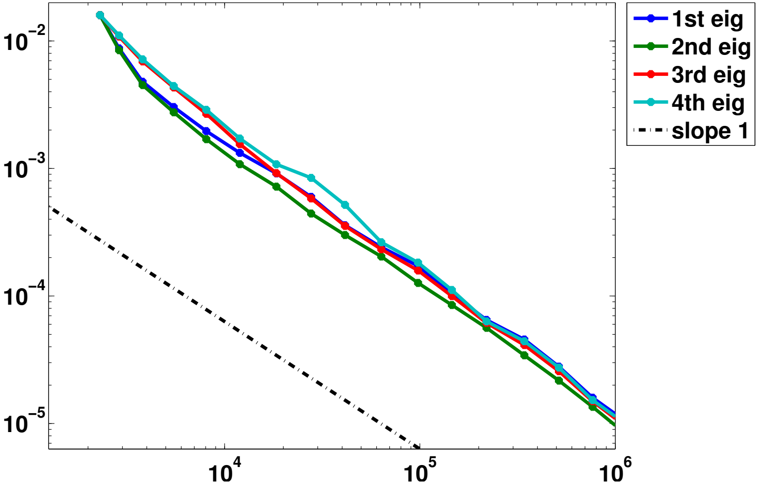
Refinement level=15



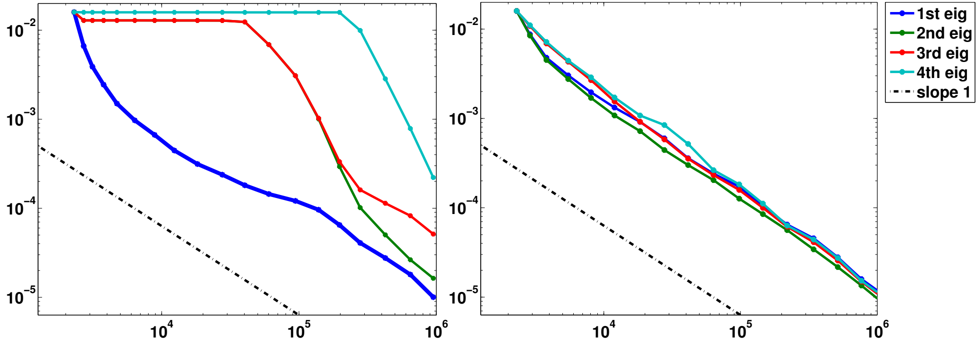


Approximation of the first four frequencies altogether

Bulk parameter=0.3, Refinement level=16



Convergence rate vs. computational cost



A tricky example

Solve the Laplace eigenvalue problem on this domain

First four exact values: $9.636\ldots, 9.638\ldots, 15.17\ldots, 15.18\ldots$
(computed on an adaptively refined mesh with 8,913,989 dofs)

The computational framework

All computations performed on a relatively old
desktop computer
(2 Intel Xeon CPU's @ 3.60GHz, 2 cores each, 16Gb RAM)

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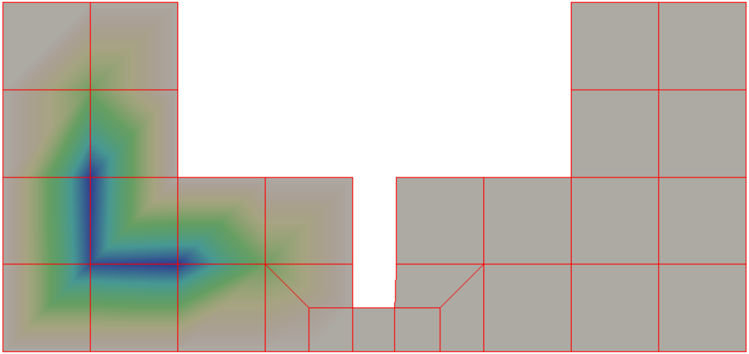
Tensor product mesh
Finite element space: continuous bilinear space Q_1
Solution of algebraic eigenvalue problem by SLEPc `http://slepc.upv.es` using `hypre`
multigrid solver
SLEPc is built on top of PETSc for the parallelization

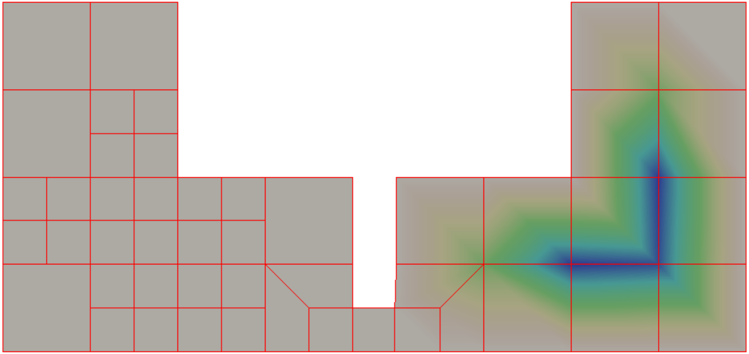
Approximation of the second frequency

Initial mesh: 48 dofs



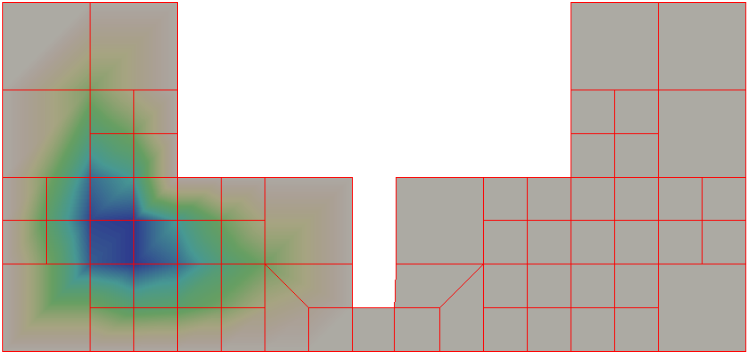
Approximation of the second frequency

Refinement # 1: 72 dofs



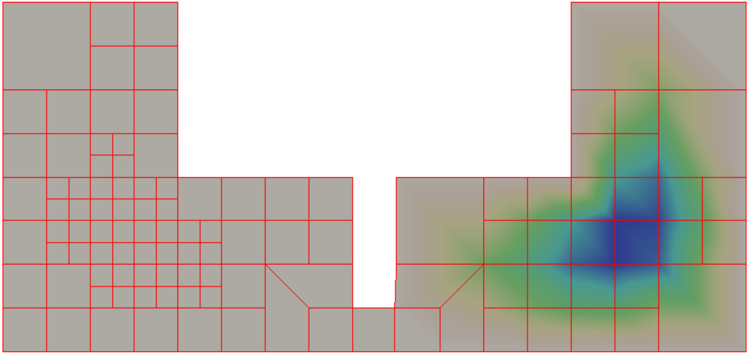
Approximation of the second frequency

Refinement # 2: 96 dofs



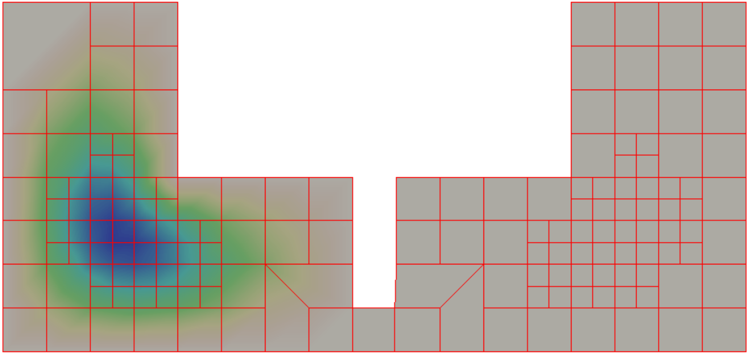
Approximation of the second frequency

Refinement # 3: 151 dofs



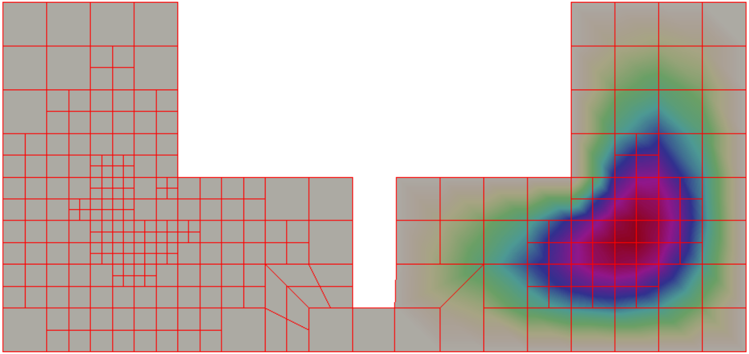
Approximation of the second frequency

Refinement # 4: 209 dofs



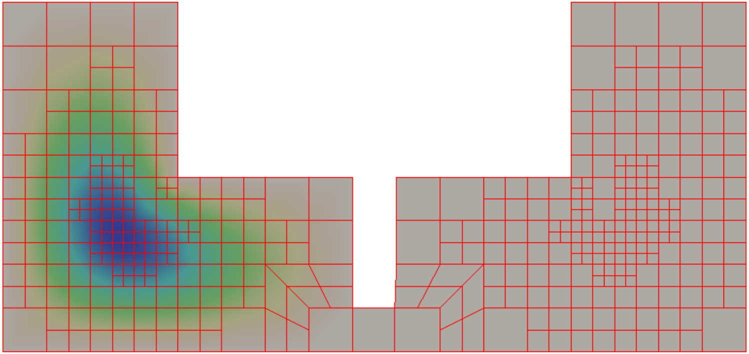
Approximation of the second frequency

Refinement # 5: 356 dofs



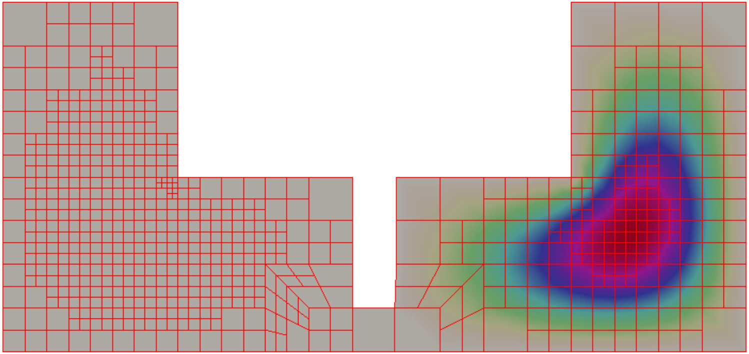
Approximation of the second frequency

Refinement # 6: 512 dofs



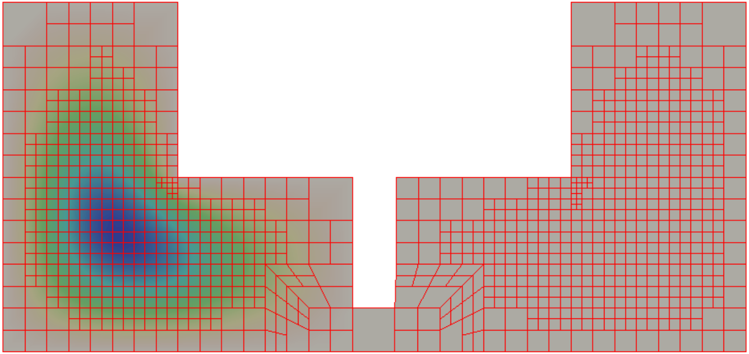
Approximation of the second frequency

Refinement # 7: 822 dofs



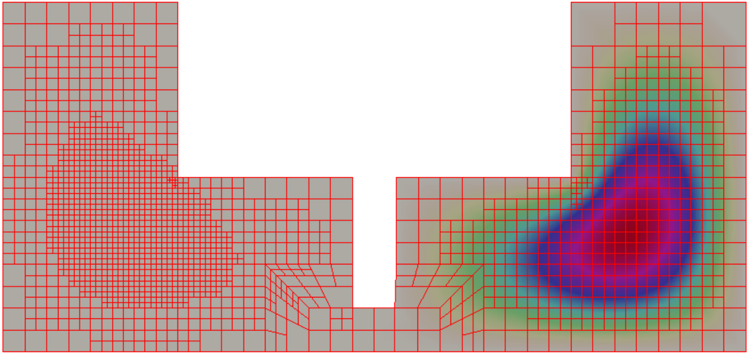
Approximation of the second frequency

Refinement # 8: 1,161 dofs



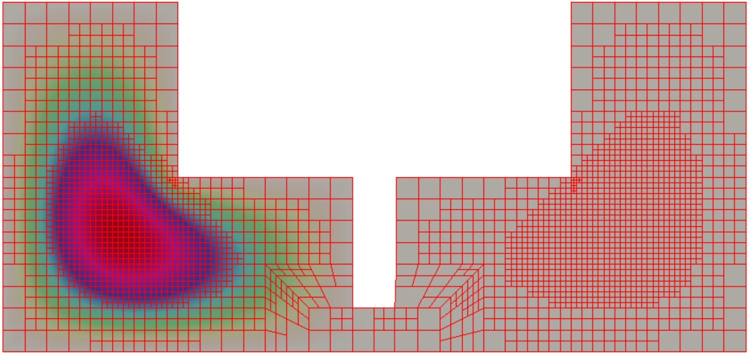
Approximation of the second frequency

Refinement # 9: 2,014 dofs



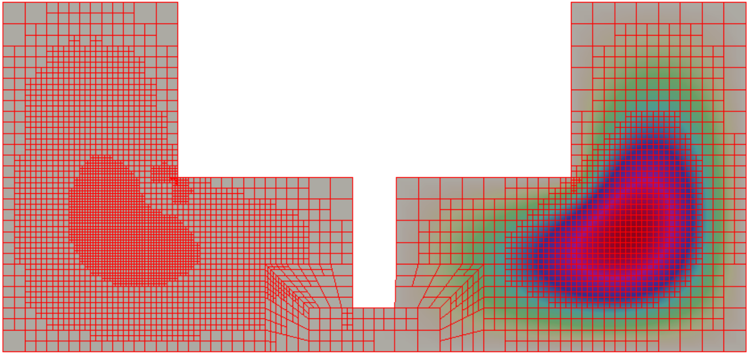
Approximation of the second frequency

Refinement # 10: 2,906 dofs



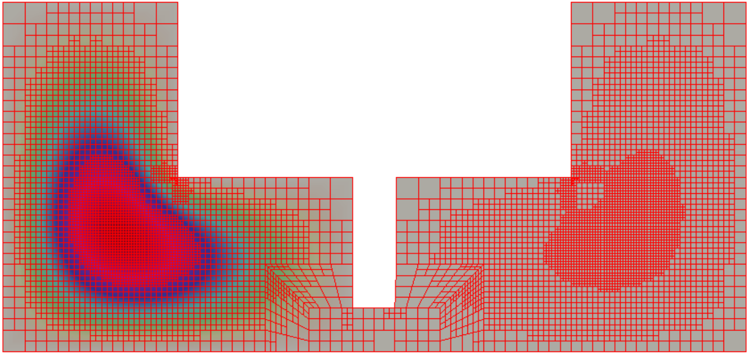
Approximation of the second frequency

Refinement # 11: 5,255 dofs



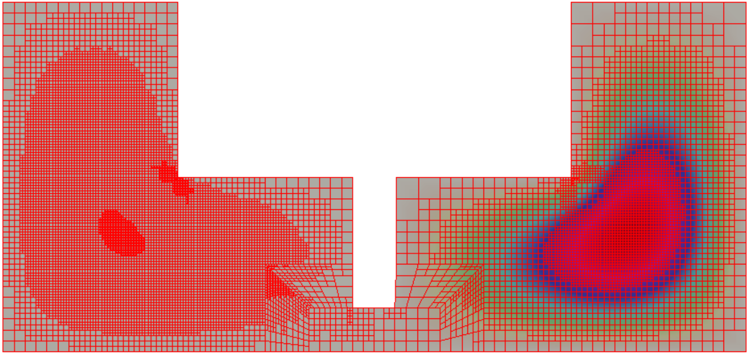
Approximation of the second frequency

Refinement # 12: 7,740 dofs



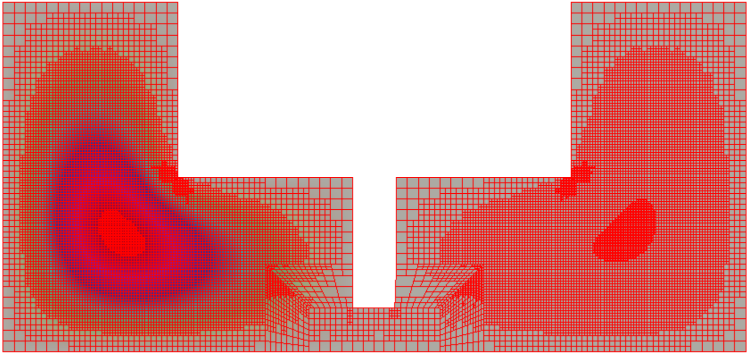
Approximation of the second frequency

Refinement # 13: 13,701 dofs



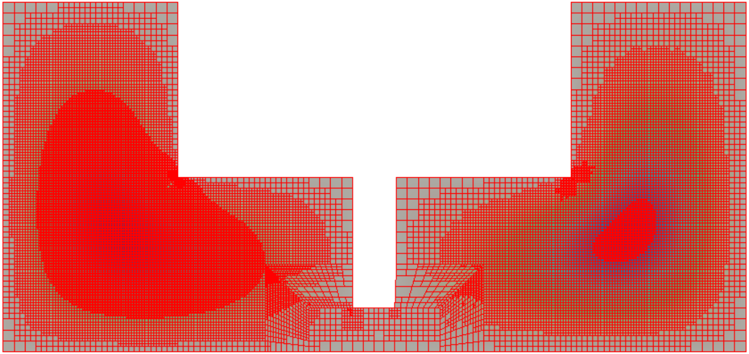
Approximation of the second frequency

Refinement # 14: 20,033 dofs



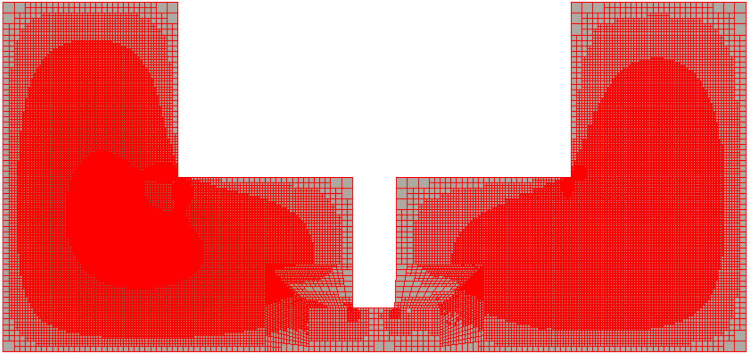
Approximation of the second frequency

Refinement # 15: 35,787 dofs



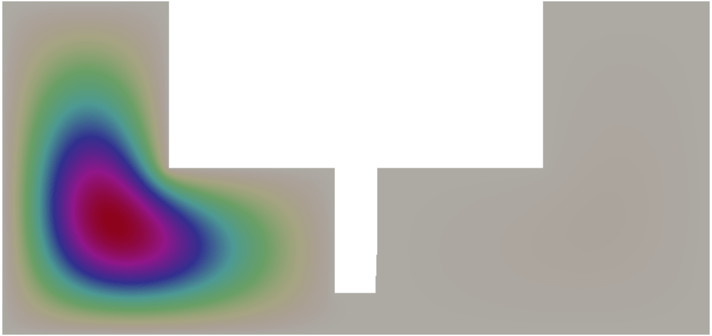
Approximation of the second frequency

Refinement # 16: 92,676 dofs



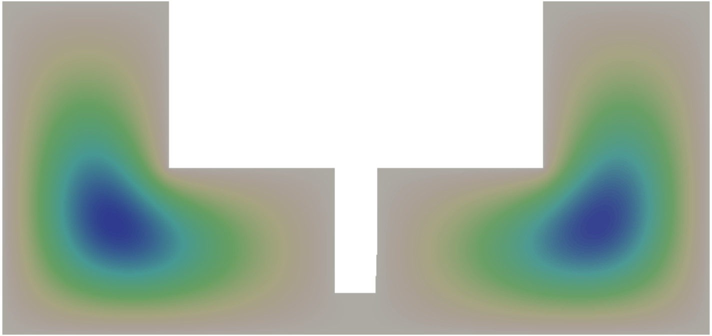
Better view

Refinement # 14: 20,033 dofs



Better view

Refinement # 15: 35,787 dofs



Better view

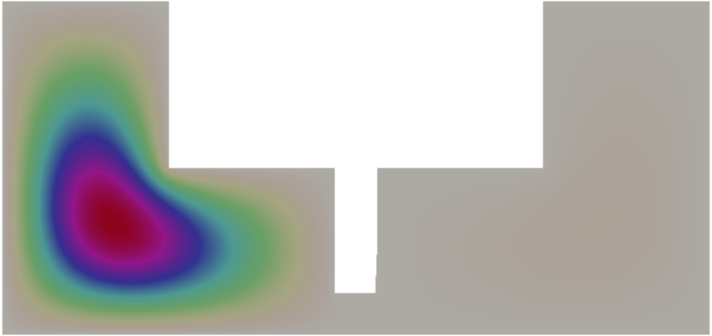
Refinement # 16: 92,676 dofs

Better view

Refinement # 17: 199,818 dofs

Better view

Refinement # 18: 485,503 dofs



Better view

Refinement # 19: 1,312,781 dofs

How much does it cost?

Global steps (Except Output)	CPU Time	
Solve	738,00	45,97%
Assemble System	337,00	20,99%
Estimate Error	239,00	14,89%
Mark and Refine	226,00	14,08%
Setup Dofs	38,00	2,37%
Create snapshot	24,90	1,55%
Setup System	2,34	0,15%
	1605,24	**100,00%**

AFEM for clusters of eigenvalues

Cluster of length N

$\lambda_{n+1}, \ldots, \lambda_{n+\mathrm{N}}$

$J = \{n+1, \ldots, n+\mathrm{N}\}$

Corresponding combination of eigenspaces

$W = \mathrm{span}\{u_j \mid j \in J\}$

$W_{\mathcal{T}_h} = W_h = \mathrm{span}\{u_{h,j} \mid j \in J\}$

AFEM for clusters of eigenvalues

Cluster of length N

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Corresponding combination of eigenspaces

$W = \mathrm{span}\{u_j \mid j \in J\}$

$W_{\mathcal{T}_h} = W_h = \mathrm{span}\{u_{h,j} \mid j \in J\}$

How to implement the AFEM scheme

Consider contribution of all elements in W_ℓ simultaneously

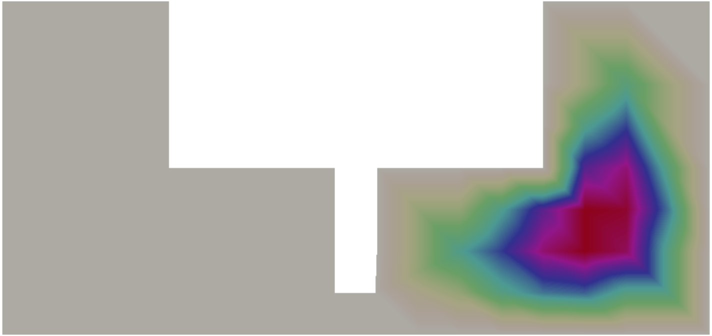
$$\theta \sum_{j \in J} \eta_{\ell,j}(\mathcal{T}_\ell)^2 \leq \sum_{j \in J} \eta_{\ell,j}(\mathcal{M}_\ell)^2$$

Approximation of the second frequency (cluster of two)

Initial mesh: 48 dofs

Approximation of the second frequency (cluster of two)

Refinement # 1: 99 dofs

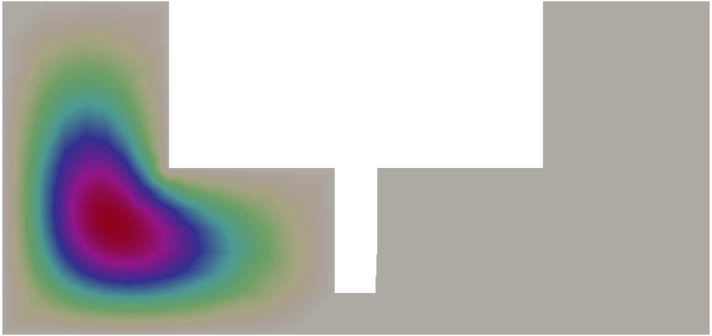


Approximation of the second frequency (cluster of two)

Refinement # 2: 221 dofs

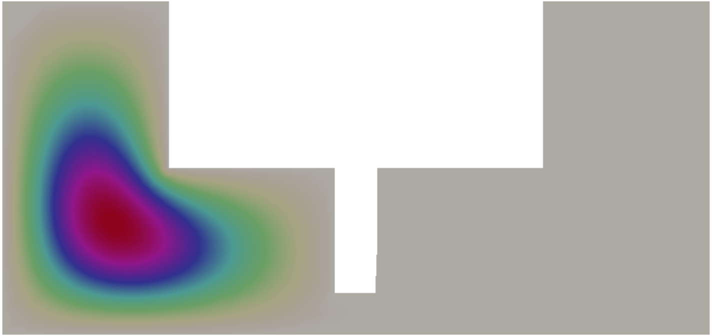
Approximation of the second frequency (cluster of two)

Refinement # 3: 543 dofs



Approximation of the second frequency (cluster of two)

Refinement # 4: 1,295 dofs



Approximation of the second frequency (cluster of two)

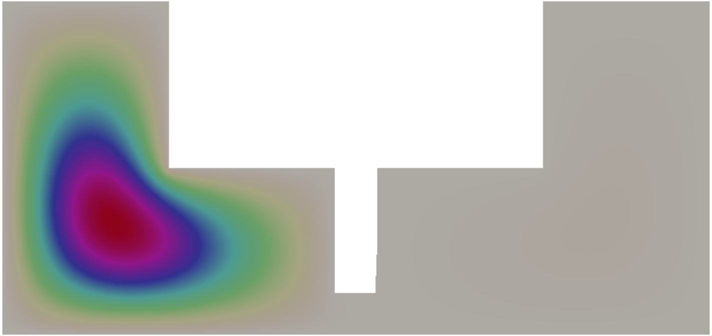
Refinement # 5: 3,141 dofs

Approximation of the second frequency (cluster of two)

Refinement # 6: 8,324 dofs

Approximation of the second frequency (cluster of two)

Refinement # 7: 21,419 dofs



Approximation of the second frequency (cluster of two)

Refinement # 8: 52,783 dofs

Approximation of the second frequency (cluster of two)

Refinement # 9: 143,641 dofs

Approximation of the second frequency (cluster of two)

Refinement # 10: 384,389 dofs

Approximation of the second frequency (cluster of two)

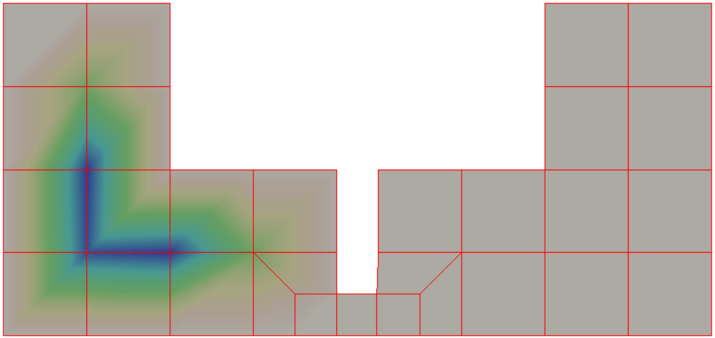
Refinement # 11: 949,442 dofs

Approximation of the second frequency (cluster of two)

Refinement # 12: 2,559,242 dofs

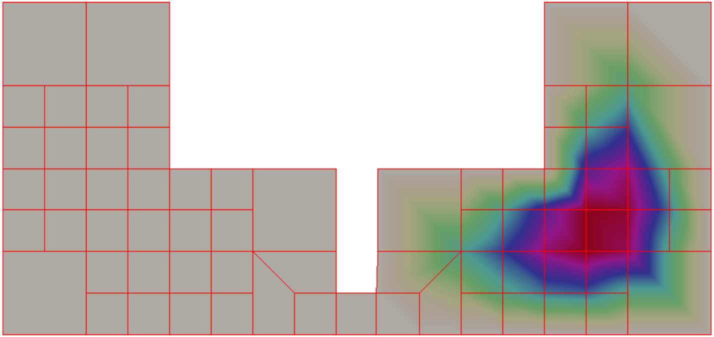
Cluster of two: underlying mesh

Initial mesh: 48 dofs



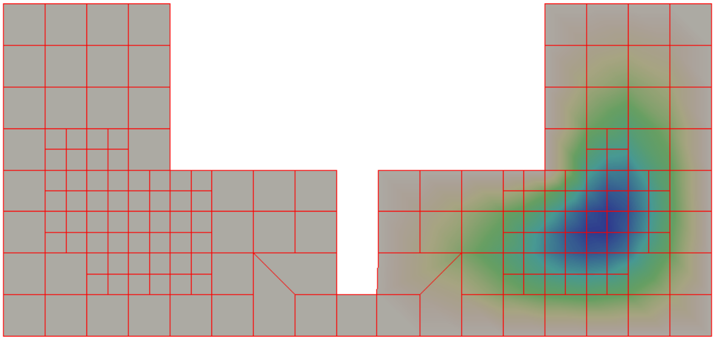
Cluster of two: underlying mesh

Refinement # 1: 99 dofs



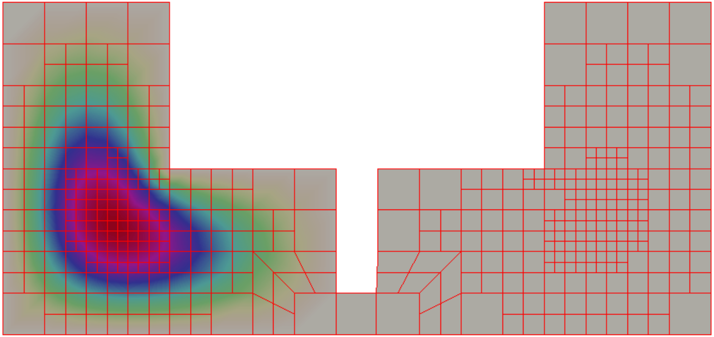
Cluster of two: underlying mesh

Refinement # 2: 221 dofs



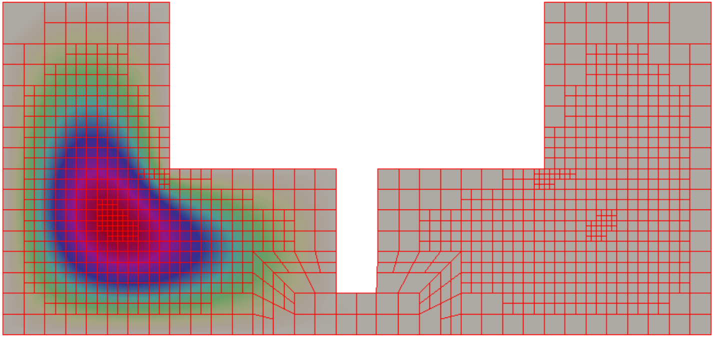
Cluster of two: underlying mesh

Refinement # 3: 543 dofs



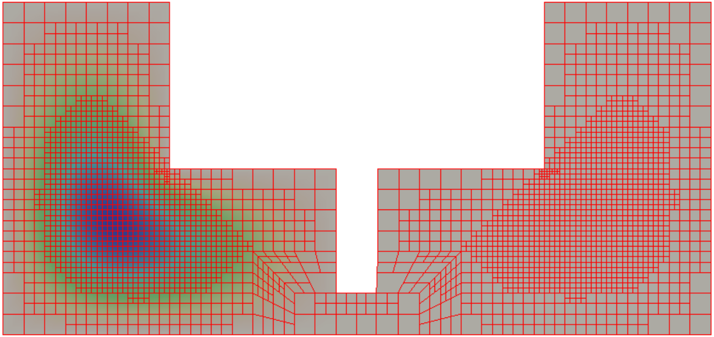
Cluster of two: underlying mesh

Refinement # 4: 1,295 dofs



Cluster of two: underlying mesh

Refinement # 5: 3,141 dofs



Cluster of two: underlying mesh

Refinement # 6: 8,324 dofs