Finite Element Approximation of Eigenvalue Problems
Lesson 3

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Buenos Aires, 29 July 2026

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Outline

Mixed approximation of Laplacian in 1D

Find $\lambda \in \mathbb{R}$ and $u \in L^2(0,\pi)$ such that for some $s \in H^1(0,\pi)$

$$\begin{cases} (s,t) + (t',u) = 0 & \forall t \in H^1(0,\pi) \quad s = u' \\ (s',v) = -\lambda(u,v) & \forall v \in L^2(0,\pi) \quad s' = -\lambda u \end{cases}$$

After conforming discretization $\Sigma_h \subset \Sigma = H^1(0,\pi)$ and $U_h \subset U = L^2(0,\pi)$ the discrete problem has the following matrix form

$$\begin{bmatrix} \mathsf{A} & \mathsf{B}^\top \\ \mathsf{B} & 0 \end{bmatrix} \begin{bmatrix} \mathsf{x} \\ \mathsf{y} \end{bmatrix} = \lambda \begin{bmatrix} 0 & 0 \\ 0 & -\mathsf{M} \end{bmatrix} \begin{bmatrix} \mathsf{x} \\ \mathsf{y} \end{bmatrix}$$

The good element

$P_1 - P_0$ scheme (in general, $P_{k+1} - P_k$)
Same eigenvalues as for the standard Galerkin P_1 scheme

$$\lambda_h^{(k)} = \frac{6}{h^2} \cdot \frac{1 - \cos kh}{2 + \cos kh}$$

$$u_h^{(k)}|_{]ih,(i+1)h[} = \frac{s_h^{(k)}(ih) - s_h^{(k)}((i+1)h)}{h\lambda_h^{(k)}}$$

$$s_h^{(k)}(ih) = k\cos(kih)$$

$$i = 0, \dots, N \quad (N = \text{number of intervals})$$

$$k = 1, \dots, N$$

A troublesome element

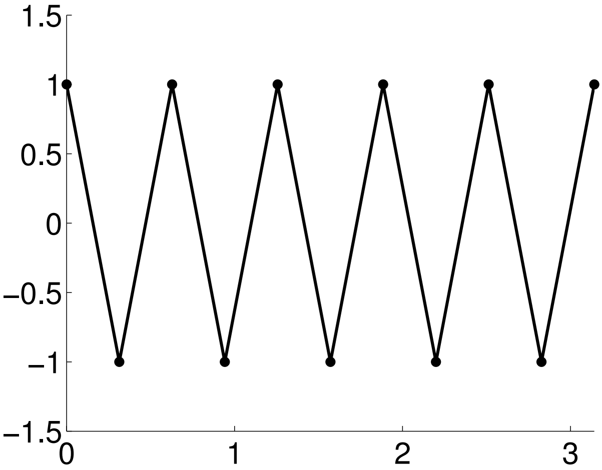
$P_1 - P_1$ scheme

	Computed eigenvalue (rate)				
	$N = 8$	$N = 16$	$N = 32$	$N = 64$	$N = 128$
0	0.0000	-0.0000	-0.0000	-0.0000	-0.0000
1	1.0001	1.0000 (4.1)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	3.9660	3.9981 (4.2)	3.9999 (4.0)	4.0000 (4.0)	4.0000 (4.0)
9	7.4257	8.5541 (1.8)	8.8854 (2.0)	8.9711 (2.0)	8.9928 (2.0)
9	8.7603	8.9873 (4.2)	8.9992 (4.1)	9.0000 (4.0)	9.0000 (4.0)
16	14.8408	15.9501 (4.5)	15.9971 (4.1)	15.9998 (4.0)	16.0000 (4.0)
25	16.7900	24.5524 (4.2)	24.9780 (4.3)	24.9987 (4.1)	24.9999 (4.0)
36	38.7154	29.7390 (-1.2)	34.2165(1.8)	35.5415 (2.0)	35.8846 (2.0)
36	39.0906	35.0393 (1.7)	35.9492 (4.2)	35.9970 (4.1)	35.9998 (4.0)
49		46.7793	48.8925 (4.4)	48.9937 (4.1)	48.9996 (4.0)

Remark: Now the eigenvalues are not always approximated from above

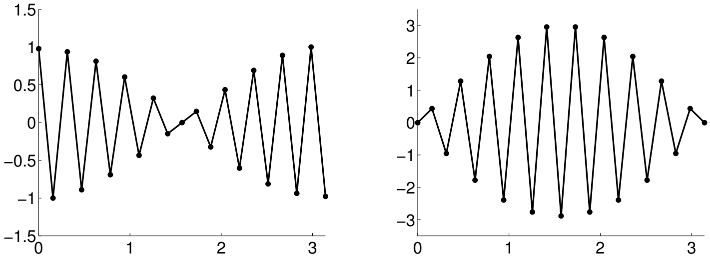
First spurious eigenfunction

$\lambda = 0$

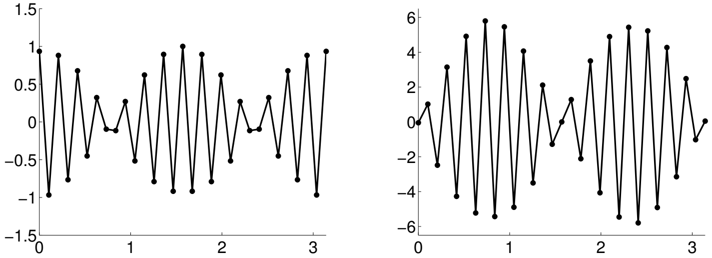


Higher order spurious eigenfunctions

$\boxed{\lambda \simeq 9}$



$\boxed{\lambda \simeq 36}$

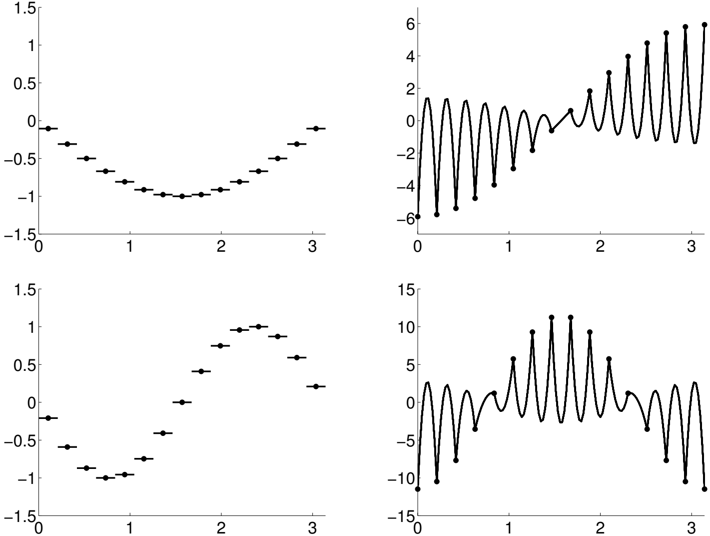


An intriguing element

$P_2 - P_0$ scheme

	Computed eigenvalue (rate with respect to 6λ)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	5.7061	5.9238 (1.9)	5.9808 (2.0)	5.9952 (2.0)	5.9988 (2.0)
4	19.8800	22.8245 (1.8)	23.6953 (1.9)	23.9231 (2.0)	23.9807 (2.0)
9	36.7065	48.3798 (1.6)	52.4809 (1.9)	53.6123 (2.0)	53.9026 (2.0)
16	51.8764	79.5201 (1.4)	91.2978 (1.8)	94.7814 (1.9)	95.6925 (2.0)
25	63.6140	113.1819 (1.2)	138.8165 (1.7)	147.0451 (1.9)	149.2506 (2.0)
36	71.6666	146.8261 (1.1)	193.5192 (1.6)	209.9235 (1.9)	214.4494 (2.0)
49	76.3051	178.6404 (0.9)	253.8044 (1.5)	282.8515 (1.9)	291.1344 (2.0)
64	77.8147	207.5058 (0.8)	318.0804 (1.4)	365.1912 (1.8)	379.1255 (1.9)
81		232.8461	384.8425 (1.3)	456.2445 (1.8)	478.2172 (1.9)
100		254.4561	452.7277 (1.2)	555.2659 (1.7)	588.1806 (1.9)
#	8	16	32	64	128

Eigenfunctions for the $P_2 - P_0$ element



Another intriguing example in 2D

Neumann eigenvalue problem for the Laplacian
Find $\lambda \in \mathbb{R}$ and $u \in L^2_0(\Omega)$ such that for some $\boldsymbol{\sigma} \in \mathbf{H}_0(\mathrm{div};\Omega)$

$$\begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathrm{div};\Omega) \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) = -\lambda(u,v) & \forall v \in L^2_0(\Omega) \end{cases}$$

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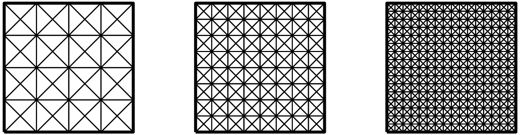
Criss-cross mesh sequence, $P_1 - \mathrm{div}(P_1)$ scheme



	Computed eigenvalue (rate)				
	$N = 2$	$N = 4$	$N = 8$	$N = 16$	$N = 32$
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
1	1.0662	1.0170 (2.0)	1.0043 (2.0)	1.0011 (2.0)	1.0003 (2.0)
2	2.2035	2.0678 (1.6)	2.0171 (2.0)	2.0043 (2.0)	2.0011 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
4	4.8634	4.2647 (1.7)	4.0680 (2.0)	4.0171 (2.0)	4.0043 (2.0)
5	6.1338	5.3971 (1.5)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
5	6.4846	5.3971 (1.9)	5.1063 (1.9)	5.0267 (2.0)	5.0067 (2.0)
6	6.4846	5.6712 (0.6)	5.9229 (2.1)	5.9807 (2.0)	5.9952 (2.0)
8	11.0924	8.8141 (1.9)	8.2713 (1.6)	8.0685 (2.0)	8.0171 (2.0)
9	11.0924	10.2540 (0.7)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
9	11.1164	10.2540 (0.8)	9.3408 (1.9)	9.0864 (2.0)	9.0217 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
10		10.9539	10.4193 (1.2)	10.1067 (2.0)	10.0268 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
13		11.1347	13.7027 (1.4)	13.1804 (2.0)	13.0452 (2.0)
15		9.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
15		19.4537	13.9639 (2.1)	14.7166 (1.9)	14.9272 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
16		19.7860	17.0588 (1.8)	16.2722 (2.0)	16.0684 (2.0)
17		20.9907	18.1813 (1.8)	17.3073 (1.9)	17.0773 (2.0)
dof	11	47	191	767	3071

Spurious modes

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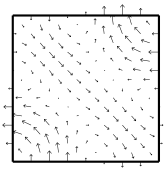

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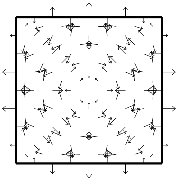

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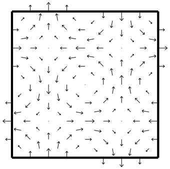
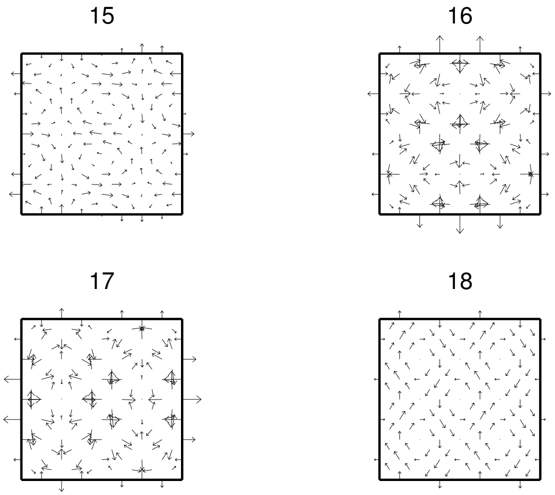


More spurious modes



Source vs. eigenvalue problem in mixed form

B.–Brezzi–Gastaldi '00

N.B.

The criss-cross $P_1 - \mathrm{div}(P_1)$ element is a good element for the *source* problem (inf-sup condition OK!)

The $Q_1 - P_0$ scheme

B.–Durán–Gastaldi '99

The discrete eigenvalues can be explicitly computed:

$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

$$\boldsymbol{\sigma}_h^{(mn)} = (\sigma_1^{(mn)}, \sigma_2^{(mn)})$$

$$\sigma_1^{(m,n)}(x_i, y_j) = \frac{2}{h}\sin\left(\frac{mh}{2}\right)\cos\left(\frac{nh}{2}\right)\sin(mx_i)\cos(ny_j)$$

$$\sigma_2^{(m,n)}(x_i, y_j) = \frac{2}{h}\cos\left(\frac{mh}{2}\right)\sin\left(\frac{nh}{2}\right)\cos(mx_i)\sin(ny_j)$$

The $Q_1 - P_0$ scheme

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$$\sigma_2^{(m,n)}(x_i, y_j) = \frac{2}{h}\cos\left(\frac{mh}{2}\right)\sin\left(\frac{nh}{2}\right)\cos(mx_i)\sin(ny_j)$$

$\boxed{\text{Does it converge?}}$

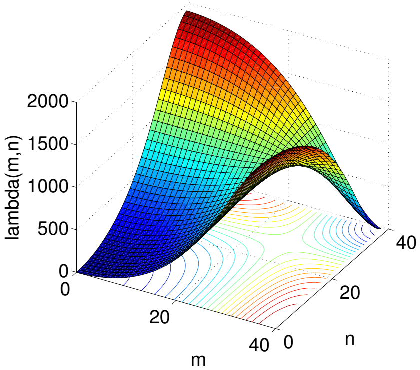
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
1	1.0524	1.0129 (2.0)	1.0032 (2.0)	1.0008 (2.0)	1.0002 (2.0)
2	1.9909	1.9995 (4.1)	2.0000 (4.0)	2.0000 (4.0)	2.0000 (4.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
4	4.8634	4.2095 (2.0)	4.0517 (2.0)	4.0129 (2.0)	4.0032 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
5	5.3896	5.1129 (1.8)	5.0288 (2.0)	5.0072 (2.0)	5.0018 (2.0)
8	7.2951	7.9636 (4.3)	7.9978 (4.1)	7.9999 (4.0)	8.0000 (4.0)
9	8.7285	10.0803 (-2.0)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
9	11.2850	10.0803 (1.1)	9.2631 (2.0)	9.0652 (2.0)	9.0163 (2.0)
10	11.2850	10.8308 (0.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
10	12.5059	10.8308 (1.6)	10.2066 (2.0)	10.0515 (2.0)	10.0129 (2.0)
13	12.5059	13.1992 (1.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
13	12.8431	13.1992 (-0.3)	13.0736 (1.4)	13.0197 (1.9)	13.0050 (2.0)
16	12.8431	14.7608 (1.3)	16.8382 (0.6)	16.2067 (2.0)	16.0515 (2.0)
16		17.5489	16.8382 (0.9)	16.2067 (2.0)	16.0515 (2.0)
17		19.4537	17.1062 (4.5)	17.1814 (-0.8)	17.0452 (2.0)
17		19.4537	17.7329 (1.7)	17.1814 (2.0)	17.0452 (2.0)
18		19.9601	17.7329 (2.9)	17.7707 (0.2)	17.9423 (2.0)
18		19.9601	17.9749 (6.3)	17.9985 (4.0)	17.9999 (4.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)
20		21.5584	20.4515 (1.8)	20.1151 (2.0)	20.0289 (2.0)

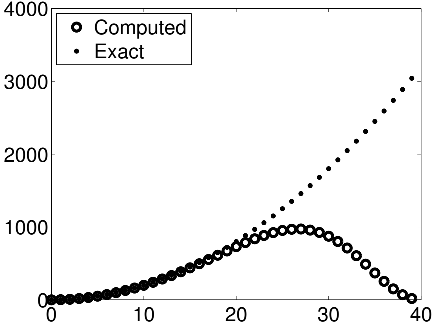
Wrong proof?

$$\lambda_h^{(mn)} = \frac{4}{h^2} \frac{\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2}) - 2\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}{1 - \frac{2}{3}(\sin^2(\frac{mh}{2}) + \sin^2(\frac{nh}{2})) + \frac{4}{9}\sin^2(\frac{mh}{2})\sin^2(\frac{nh}{2})}$$

Indeed, if $h = \pi/N$, we have:

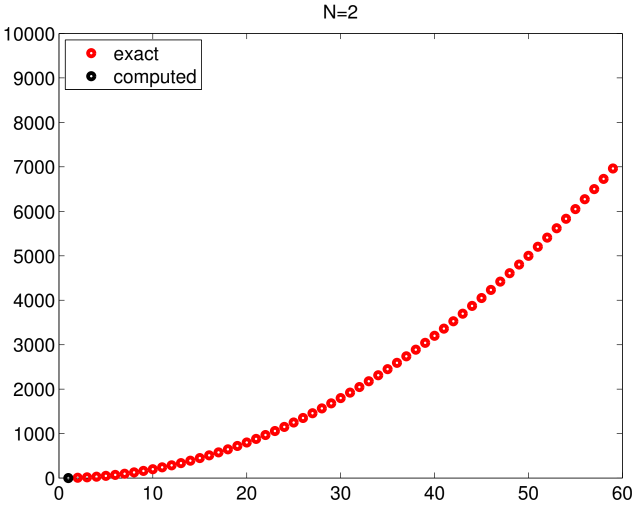
$$\lim_{N\to\infty} \lambda_h^{(N-1,N-1)} = 18$$


lambda(m,n)
2000
1500
1000
500
0
0
20
40
m
0
20
40
n

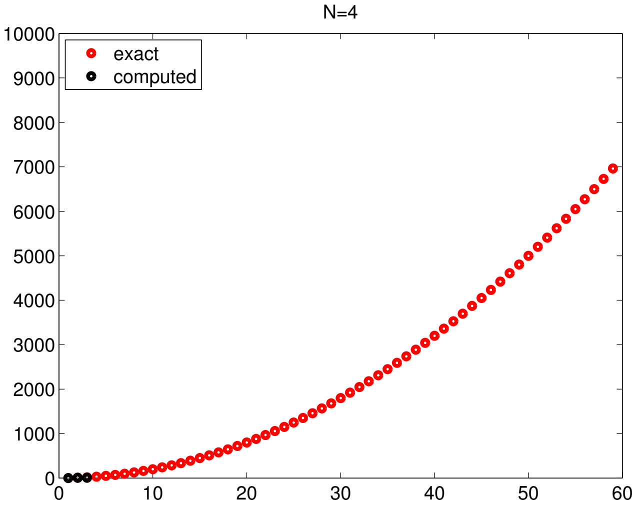

Computed
Exact
4000
3000
2000
1000
0
0
10
20
30
40

Plot for $m = n$

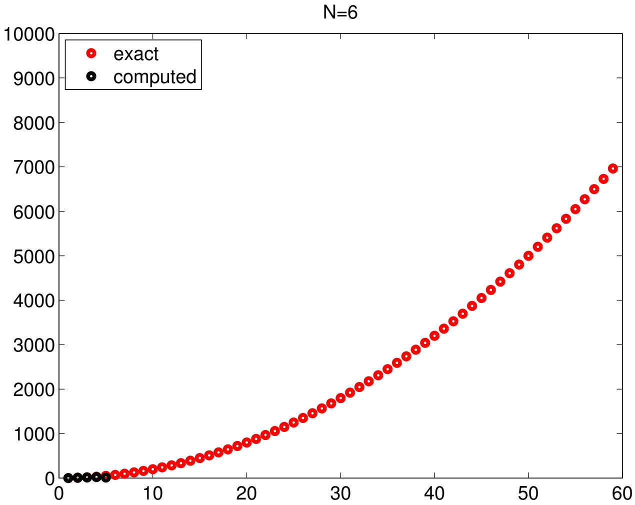
Pointwise vs. uniform convergence



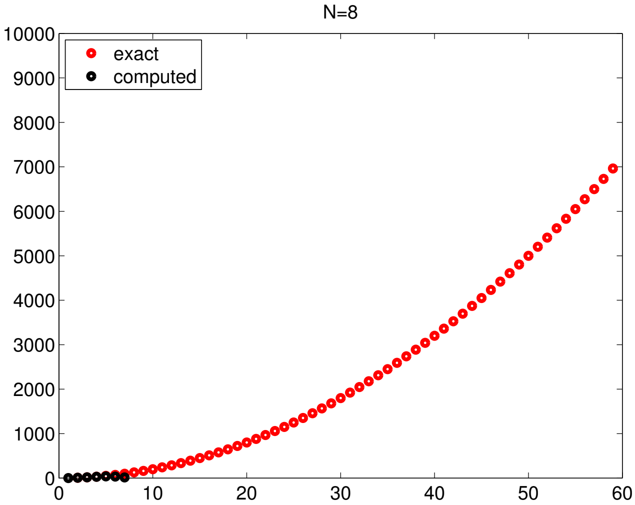
Pointwise vs. uniform convergence



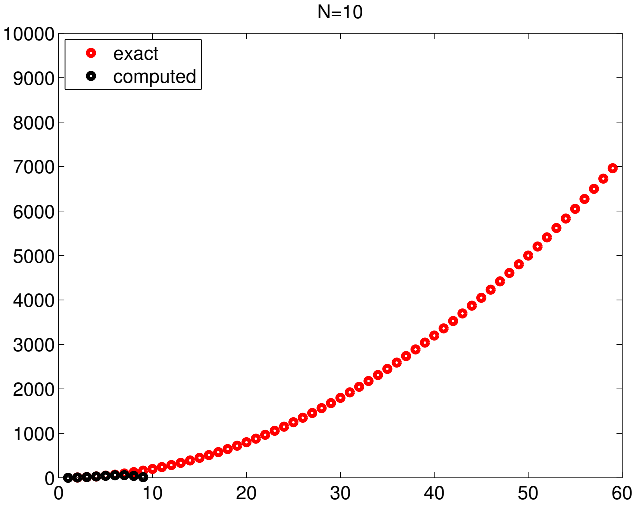
Pointwise vs. uniform convergence



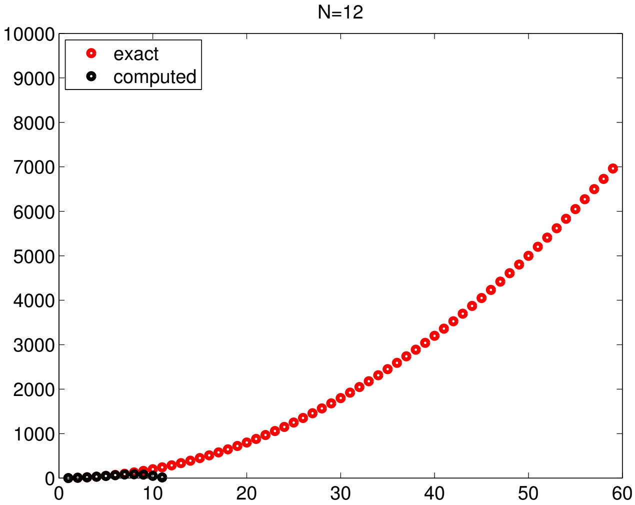
Pointwise vs. uniform convergence



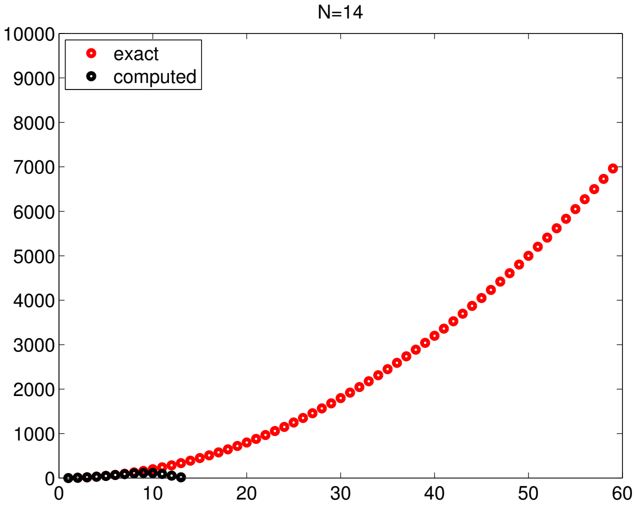
Pointwise vs. uniform convergence



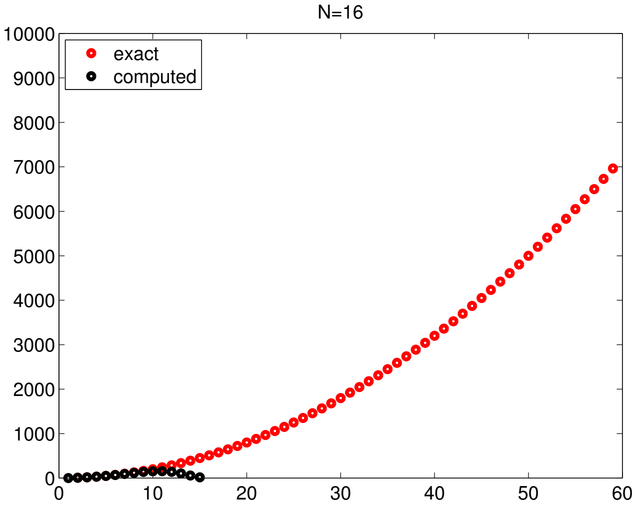
Pointwise vs. uniform convergence



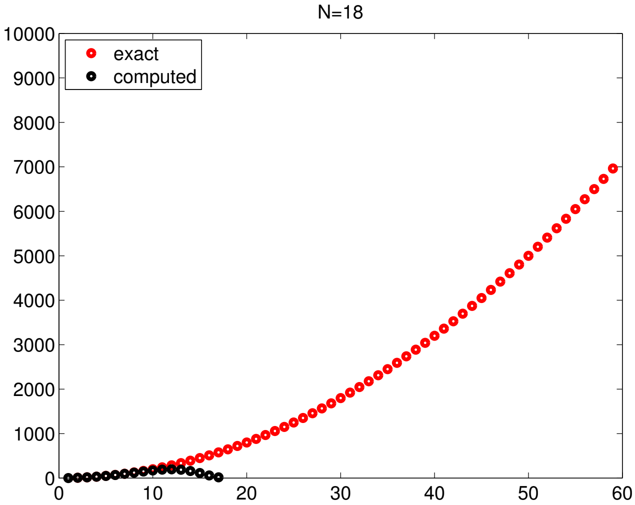
Pointwise vs. uniform convergence



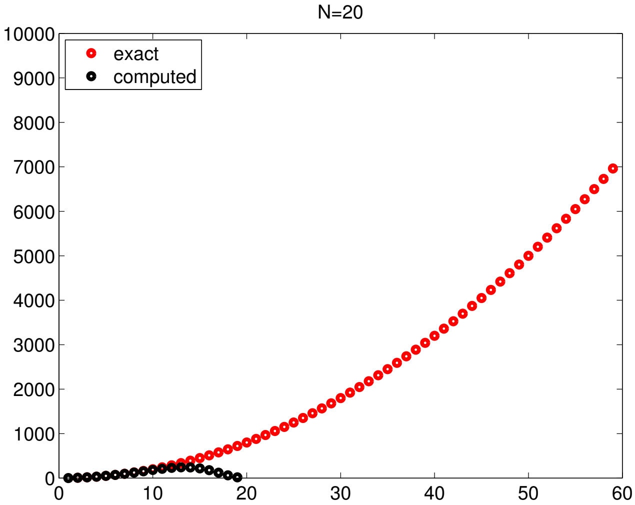
Pointwise vs. uniform convergence



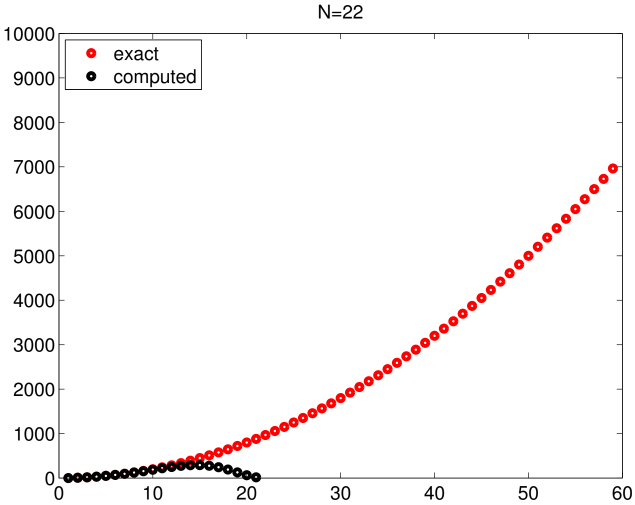
Pointwise vs. uniform convergence



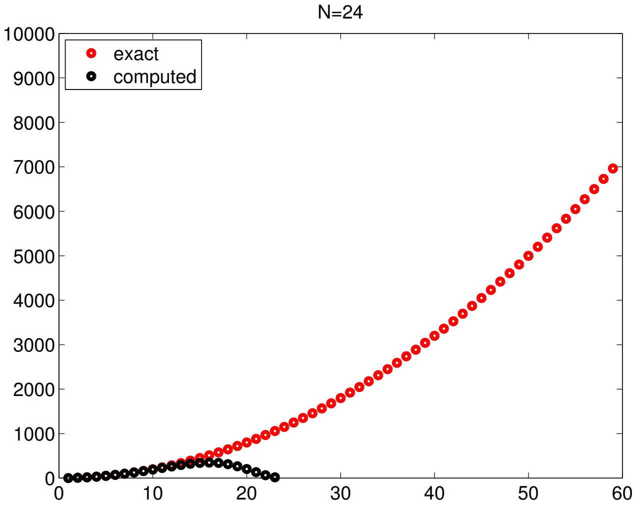
Pointwise vs. uniform convergence



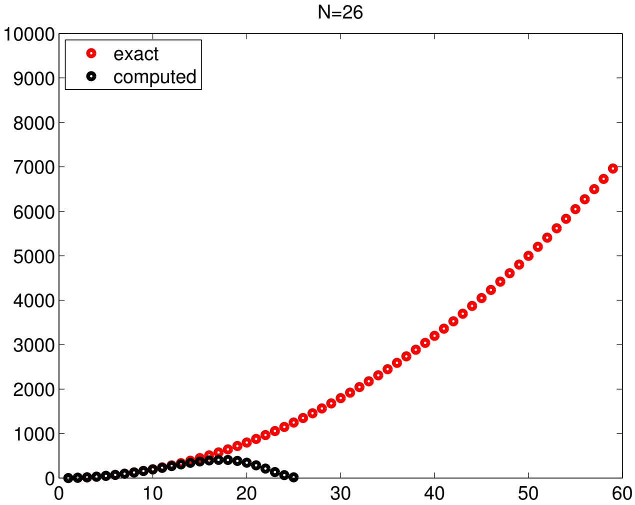
Pointwise vs. uniform convergence



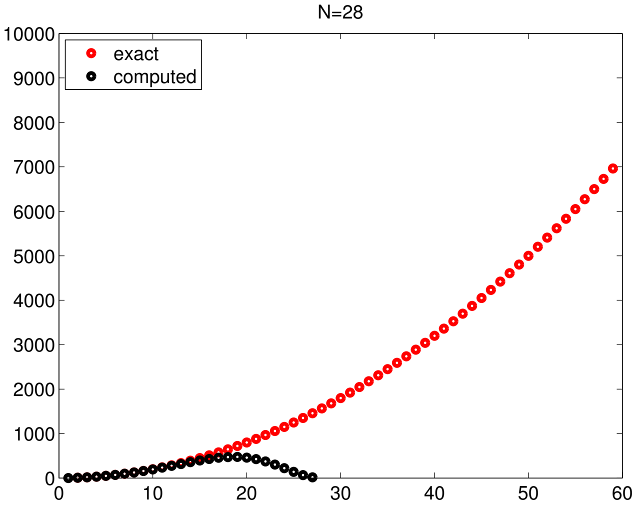
Pointwise vs. uniform convergence



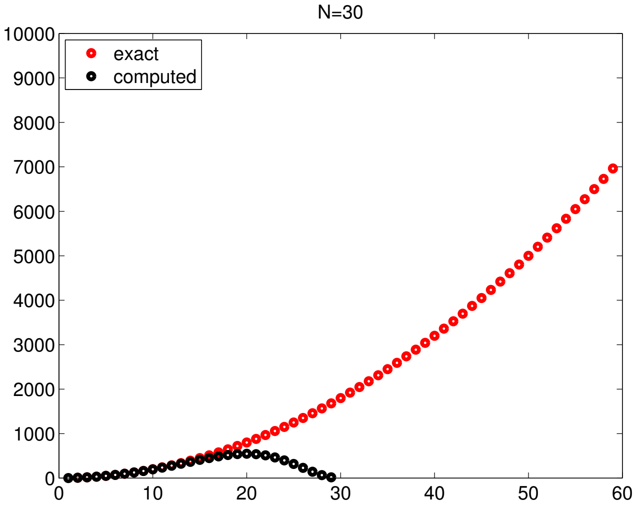
Pointwise vs. uniform convergence



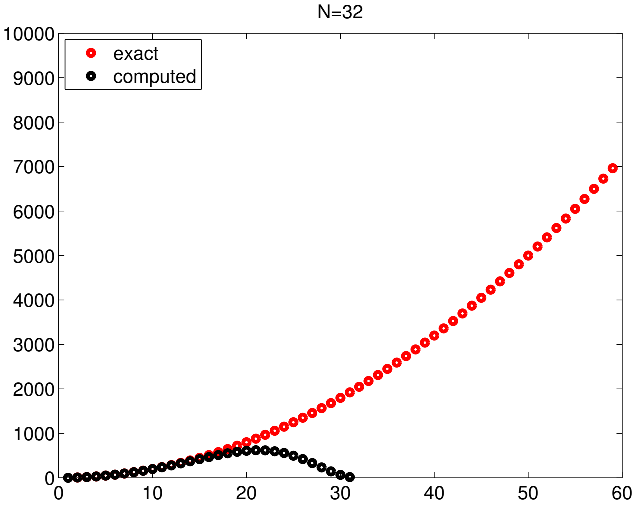
Pointwise vs. uniform convergence



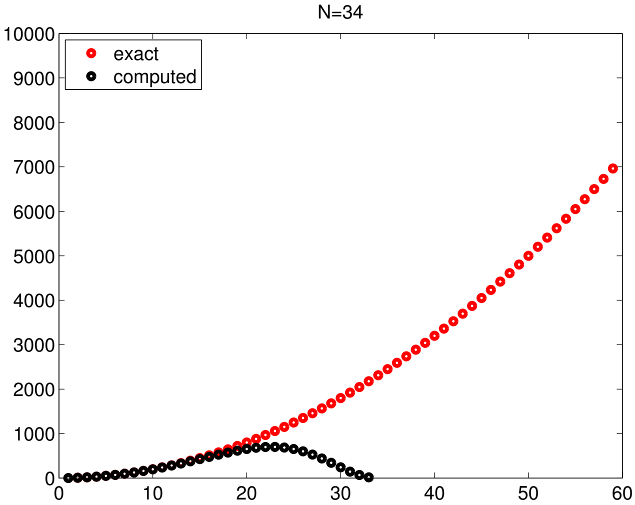
Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



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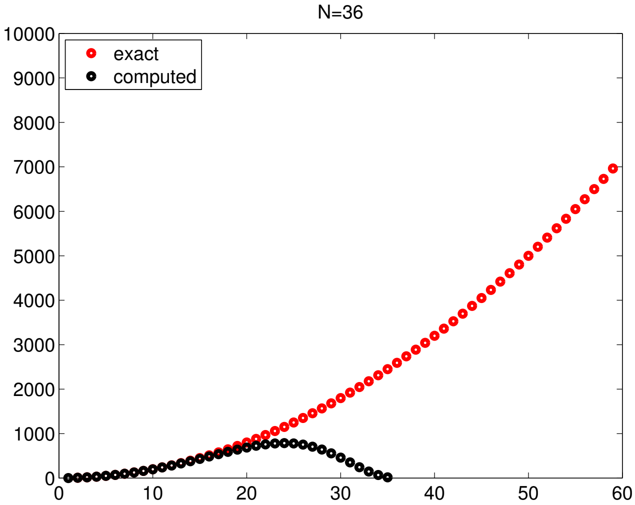
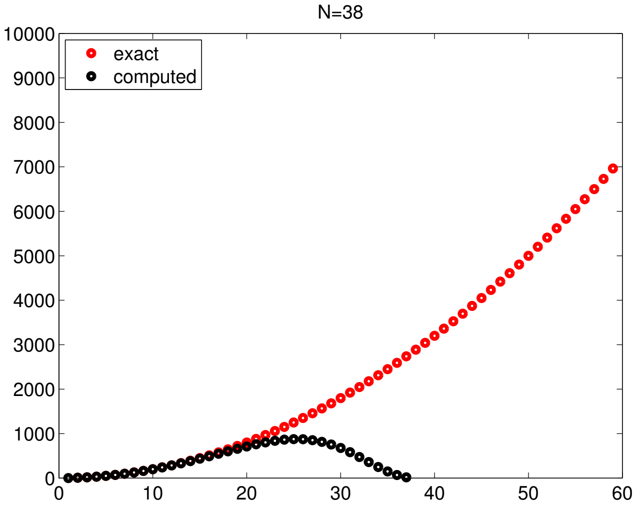


Pointwise vs. uniform convergence



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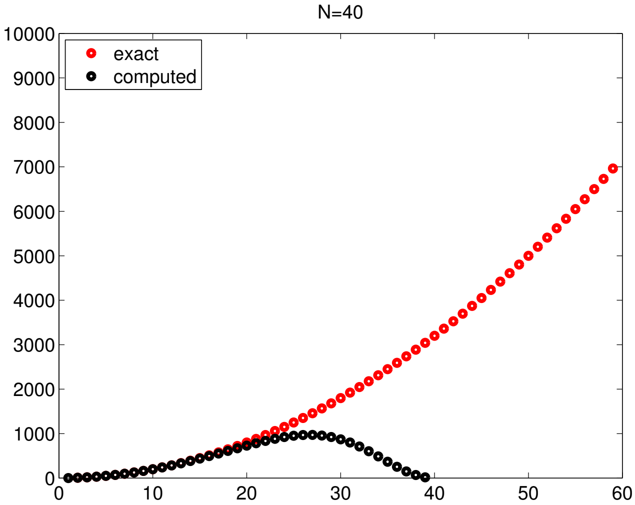

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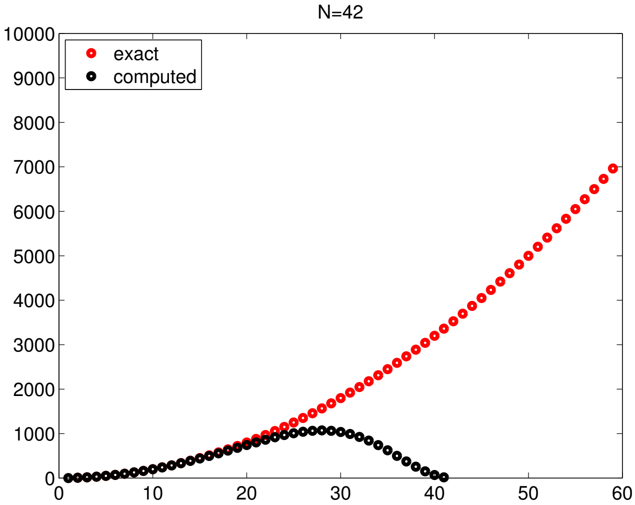
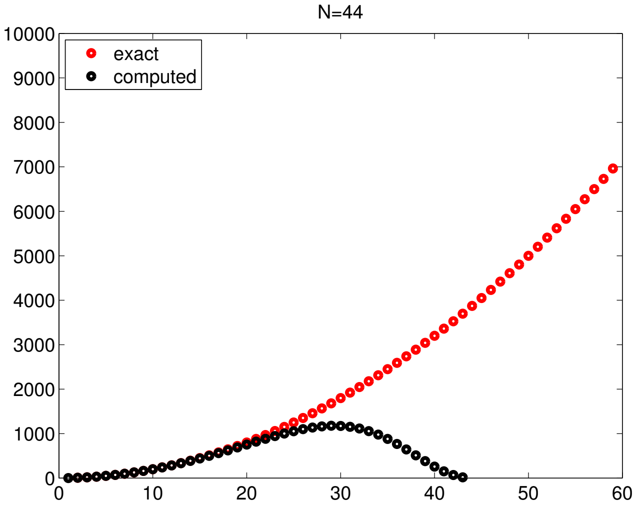


Pointwise vs. uniform convergence



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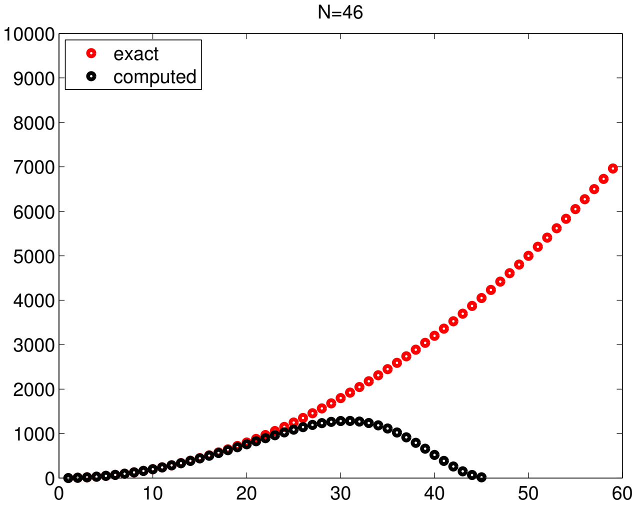
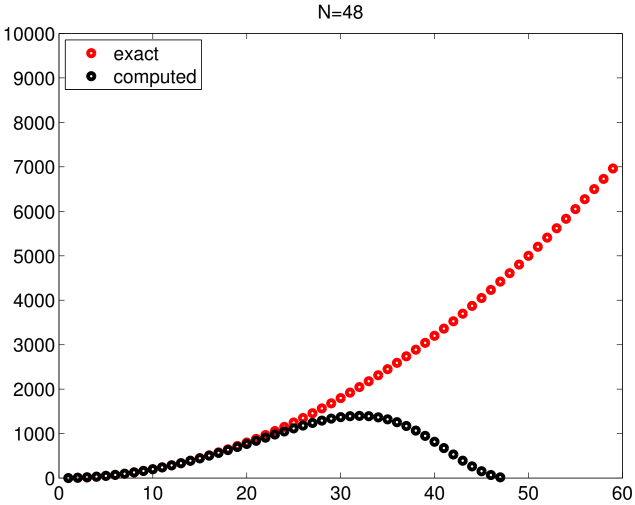
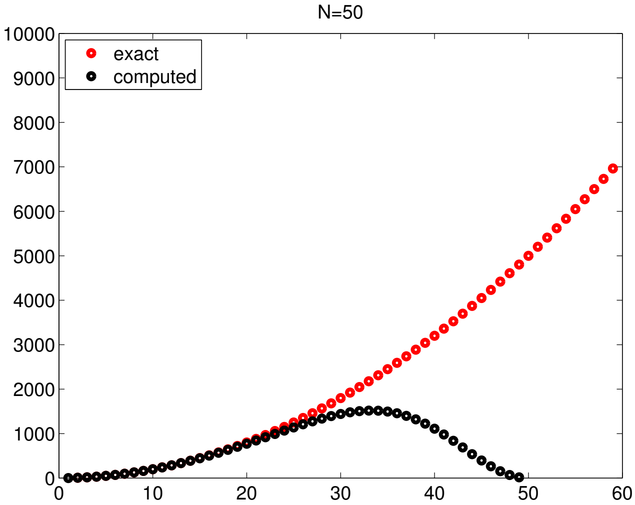


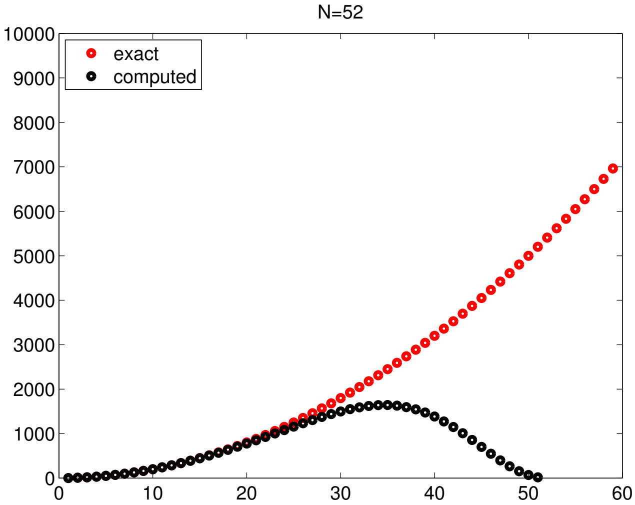
Pointwise vs. uniform convergence



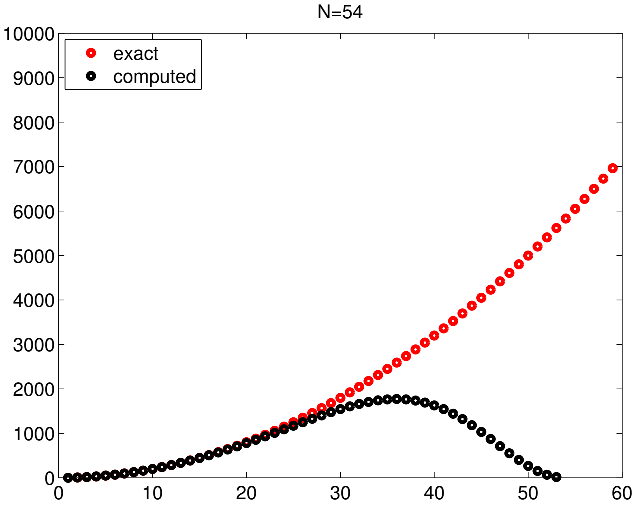
Pointwise vs. uniform convergence



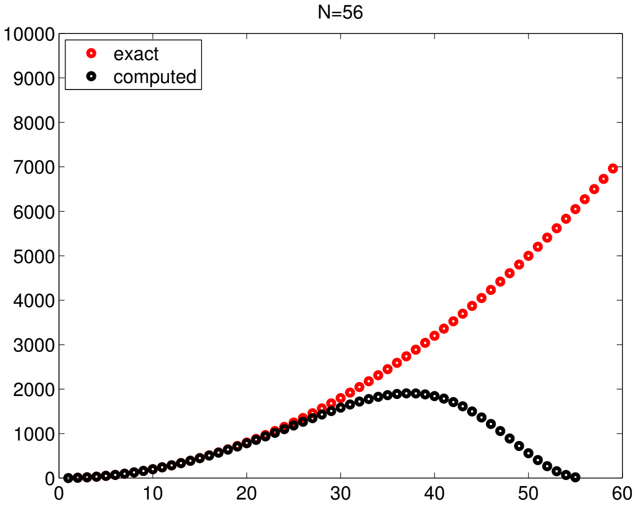
Pointwise vs. uniform convergence



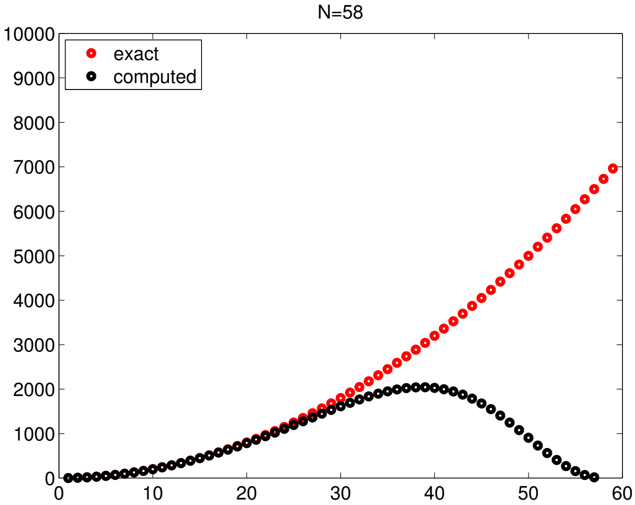
Pointwise vs. uniform convergence



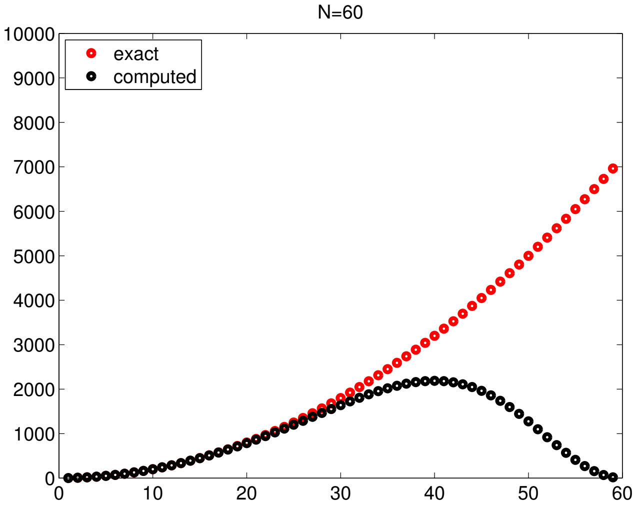
Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



Pointwise vs. uniform convergence



Raviart–Thomas element

$RT_0 - P_0$

Unstructured mesh

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.0138	1.9989 (3.6)	1.9997 (1.7)	1.9999 (2.7)	2.0000 (2.8)
5	4.8696	4.9920 (4.0)	5.0000 (8.0)	4.9999 (-2.1)	5.0000 (3.7)
5	4.8868	4.9952 (4.5)	5.0006 (3.0)	5.0000 (5.8)	5.0000 (2.6)
8	8.6905	7.9962 (7.5)	7.9974 (0.6)	7.9995 (2.5)	7.9999 (2.2)
10	9.7590	9.9725 (3.1)	9.9980 (3.8)	9.9992 (1.3)	9.9999 (3.2)
10	11.4906	9.9911 (7.4)	10.0007 (3.7)	10.0005 (0.4)	10.0001 (2.4)
13	11.9051	12.9250 (3.9)	12.9917 (3.2)	12.9998 (5.4)	12.9999 (1.8)
13	12.7210	12.9631 (2.9)	12.9950 (2.9)	13.0000 (7.5)	13.0000 (1.1)
17	13.5604	16.8450 (4.5)	16.9848 (3.4)	16.9992 (4.3)	16.9999 (2.5)
17	14.1813	16.9659 (6.4)	16.9946 (2.7)	17.0009 (2.6)	17.0000 (5.5)
#	32	142	576	2338	9400

Raviart–Thomas element (cont'ed)

Uniform mesh

	Computed (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.1048	2.0258 (2.0)	2.0064 (2.0)	2.0016 (2.0)	2.0004 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
5	5.9158	5.2225 (2.0)	5.0549 (2.0)	5.0137 (2.0)	5.0034 (2.0)
8	9.7268	8.4191 (2.0)	8.1033 (2.0)	8.0257 (2.0)	8.0064 (2.0)
10	13.8955	11.0932 (1.8)	10.2663 (2.0)	10.0660 (2.0)	10.0165 (2.0)
10	13.8955	11.0932 (1.8)	10.2663 (2.0)	10.0660 (2.0)	10.0165 (2.0)
13	17.7065	14.2898 (1.9)	13.3148 (2.0)	13.0781 (2.0)	13.0195 (2.0)
13	17.7065	14.2898 (1.9)	13.3148 (2.0)	13.0781 (2.0)	13.0195 (2.0)
17	20.5061	20.1606 (0.1)	17.8414 (1.9)	17.2075 (2.0)	17.0517 (2.0)
17	20.5061	20.4666 (0.0)	17.8414 (2.0)	17.2075 (2.0)	17.0517 (2.0)
#	16	64	256	1024	4096

Outline

Laplace eigenproblem in mixed form

Set:

- $\mathbf{\Sigma} = \mathbf{H}(\mathrm{div};\Omega)$, $U = L^2(\Omega)$ for Dirichlet boundary conditions
- $\mathbf{\Sigma} = \mathbf{H}_0(\mathrm{div};\Omega)$, $U = L^2_0(\Omega)$ for Neumann boundary conditions

Eigenvalue problem for the Laplacian

Find $\lambda \in \mathbb{R}$ and $u \in U$ with $u \neq 0$ such that for some $\boldsymbol{\sigma} \in \mathbf{\Sigma}$

$$\begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \mathbf{\Sigma} \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) = -\lambda(u,v) & \forall v \in U \end{cases}$$

Associated source problem Given $g \in L^2(\Omega)$, find $(\boldsymbol{\sigma}, u) \in \mathbf{\Sigma} \times U$ such that

$$\begin{cases} (\boldsymbol{\sigma},\boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) = 0 & \forall \boldsymbol{\tau} \in \mathbf{\Sigma} \\ (\mathrm{div}\,\boldsymbol{\sigma}, v) = -(g,v) & \forall v \in U \end{cases}$$

Definition of the solution operator

A first natural (but wrong) definition

$$T_1 : U \to \boldsymbol{\Sigma} \times U$$
$$T_1(g) = (\boldsymbol{\sigma}, u)$$

One would like to compute eigenvalues...

Definition of the solution operator

A first natural (but wrong) definition

$$
\begin{aligned}
&T_1 : U \to \boldsymbol{\Sigma} \times U\\
&T_1(g) = (\boldsymbol{\sigma}, u)
\end{aligned}
$$

One would like to compute eigenvalues...

Second possible definition

$$
\begin{aligned}
&T_2 : \Sigma \times U \to \Sigma \times U\\
&T_2(f, g) = (\sigma, u)
\end{aligned}
$$

with

$$
\begin{cases}
(\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathrm{div}\,\boldsymbol{\tau}, u) = (f, v) & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}\\
(\mathrm{div}\,\boldsymbol{\sigma}, v) = -(g, v) & \forall v \in U
\end{cases}
$$

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with

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u) = (f, v) & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma} \\ (\operatorname{div} \boldsymbol{\sigma}, v) = -(g, v) & \forall v \in U \end{cases}$$

$$T_{\Sigma U} \boxed{\begin{array}{c} (f, g) \xmapsto{\text{cutoff}} (0, g) \xmapsto{T_2} (\sigma, u) \\ L^2 \times L^2 \longrightarrow L^2 \times L^2 \end{array}} \text{ is compact}$$

Discrete eigenvalue problem in mixed form

Let $\boldsymbol{\Sigma}_h \subset \boldsymbol{\Sigma}$, $U_h \subset U$ be finite dimensional subspaces

Discrete eigenvalue problem for the Laplacian

Find $\lambda_h \in \mathbb{R}$ and $u_h \in U_h$ with $u_h \neq 0$ such that for some $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h$

$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u_h) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \\ (\operatorname{div} \boldsymbol{\sigma}_h, v) = -\lambda(u_h, v) & \forall v \in U_h \end{cases}$$

Associated source problem Given $g \in L^2(\Omega)$, find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times U_h$ such that

$$\begin{cases} (\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, u_h) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \\ (\operatorname{div} \boldsymbol{\sigma}_h, v) = -(g, v) & \forall v \in U_h \end{cases}$$

Saddle point problem theory

We set $\mathbb{K}_h = \{\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h : (\operatorname{div} \boldsymbol{\sigma}_h, v) = 0 \quad \forall v \in U_h\}$

Ellipticity on the kernel

There exists $\alpha > 0$ such that

$$(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \alpha \|\boldsymbol{\tau}_h\|_\Sigma \quad \forall \boldsymbol{\tau}_h \in \mathbb{K}_h$$

Inf-sup condition

Fortin operator

$\Pi_h : \boldsymbol{\Sigma}^+ \to \boldsymbol{\Sigma}_h$ such that

$$\begin{aligned} &(\operatorname{div}(\boldsymbol{\sigma} - \Pi_h \boldsymbol{\sigma}), v) = 0 \quad \forall v \in U_h \\ &||\Pi_h \boldsymbol{\sigma}||_\Sigma \leq C ||\boldsymbol{\sigma}||_{\Sigma^+} \end{aligned}$$

Then we have for the source problem the following error estiamates:

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_\Sigma + ||u - u_h||_U \leq C \inf_{\boldsymbol{\tau}_h, v_h} \left(||\boldsymbol{\sigma} - \boldsymbol{\tau}_h||_\Sigma + ||u - v_h||_U \right)$$

Uniform convergence?

Let's try to follow Kolata's argument

- $T : H \to V$ is compact
- Π_h converges strongly to the identity operator from V to H

$$\boxed{\Rightarrow \quad \lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(H)} = 0}$$

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$$T_{\Sigma U} - T_{\Sigma U,h} = (I - Q_h) T_{\Sigma U}$$

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✓ $||(I - Q_h)(\sigma, u)||_{\Sigma\times U} \to 0$ for all $(\boldsymbol{\sigma}, u) \in \boldsymbol{\Sigma} \times U$

✗ $T_{\Sigma U} : L^2 \times L^2 \to \Sigma \times U$ is not compact

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Remark: standard mixed estimates don't help

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{\Sigma} + ||u - u_h||_U \le C \inf_{\boldsymbol{\tau}_h, v_h} (\underbrace{||\boldsymbol{\sigma} - \boldsymbol{\tau}_h||_{\Sigma}}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)})$$

A counterexample

Fix–Gunzburger–Nicolaides '81
Boffi-Brezzi-Gastaldi 2000

Ω square of side π
Mesh of subsquares divided into 4 triangles (crisscross)

$$\mathbf{\Sigma}_h = \{\text{continuous p.w. linears (componentwise)}\}$$
$$U_h = \operatorname{div} \mathbf{\Sigma}_h \subset \{\text{p.w. constants}\}$$

Theorem (compatibility)

For all $g \in L^2(\Omega)$ there exists a unique solution $(\boldsymbol{\sigma}_h, u_h) \in \mathbf{\Sigma}_h \times U_h$ such that

$$||\sigma - \sigma_h||_\Sigma + ||u - u_h||_U \to 0$$

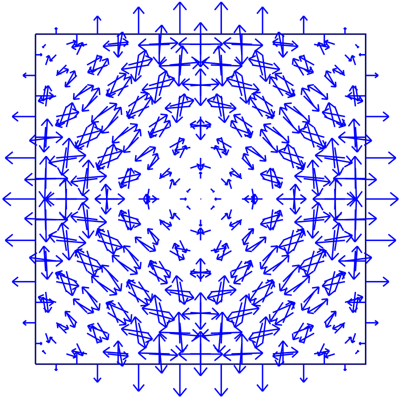
That is $||(I - Q_h)(\sigma, u)||_{\Sigma \times U} \to 0$ pointwise (inf-sup condition)

A counterexample (cont'ed)

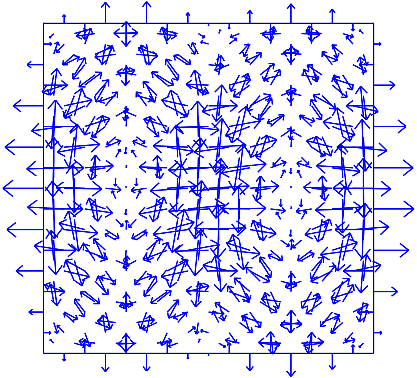
exact	computed			
1.00000	1.00428	1.00190	1.00107	1.00068
1.00000	1.00428	1.00190	1.00107	1.00069
2.00000	2.01711	2.00761	2.00428	2.00274
4.00000	4.06804	4.03037	4.01710	4.01095
4.00000	4.06804	4.03037	4.01710	4.01095
5.00000	5.10634	5.04748	5.02674	5.01712
5.00000	5.10634	5.04748	5.02674	5.01712
	5.92293	5.96578	5.98074	5.98767
8.00000	8.27128	8.12151	8.06845	8.04383
9.00000	9.34085	9.15309	9.08640	9.05537
9.00000	9.34085	9.15309	9.08640	9.05537
d.o.f.	254	574	1022	1598

Spurious eigenfunctions

1st spurious eigenfunction



2nd spurious eigenfunction



Definition of the resolvent operator (cont'ed)

A better definition

$T_U : U \to U \quad \boldsymbol{\sigma} \in \boldsymbol{\Sigma}, \quad T_U g \in U$ such that

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\operatorname{div} \boldsymbol{\tau}, T_U g) = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma} \\ (\operatorname{div} \boldsymbol{\sigma}, v) = -(g, v) & \forall v \in U \end{cases}$$

Remark: operator is now compact, but standard mixed estimates don't help again

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_\Sigma + ||u - u_h||_U \le C \inf_{\boldsymbol{\tau}_h, v_h} (\underbrace{||\boldsymbol{\sigma} - \boldsymbol{\tau}_h||_\Sigma}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)})$$

Definition of the resolvent operator (cont'ed)

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$T_U : U \to U \quad \boldsymbol{\sigma} \in \boldsymbol{\Sigma}, \quad T_U g \in U$ such that

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$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_\Sigma + ||u - u_h||_U \le C \inf_{\boldsymbol{\tau}_h, v_h} (\underbrace{||\boldsymbol{\sigma} - \boldsymbol{\tau}_h||_\Sigma}_{O(1)} + \underbrace{||u - v_h||_U}_{O(h)})$$

Remark: we need an estimate for u_h which does not involve $\operatorname{div} \boldsymbol{\sigma}$

Toward the proof of the uniform convergence

Let $g \in L^2(\Omega)(= U)$ be given. Assume the ellipticity in the kernel and the existence of a Fortin operator, then we have:

$$\begin{aligned} ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{L^2} &\leq C\left(||\boldsymbol{\sigma} - \Pi_h\boldsymbol{\sigma}||_{L^2} + (1/\sqrt{\alpha}) \inf_{v_h \in U_h} ||T_U g - v_h||_U\right) \\ ||\mathcal{T}_U g - \mathcal{T}_{U,h} g||_U &\leq C\left(\inf_{v_h \in U_h} ||T_U g - v_h||_U + ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{L^2}\right) \end{aligned}$$

Proof

$P = L^2$-projection onto U_h, $\Pi_h : \mathbf{\Sigma}^+ \to \mathbf{\Sigma}_h$ Fortin operator

$$
\begin{aligned}
\|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}^2 &= (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \\
&= (\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}, \Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - (\operatorname{div}(\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h), u - Pu) \\
&\le \|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{L^2} \|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} + \|\operatorname{div}(\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2} \|u - Pu\|_U \\
&\le \|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} \left(\|\Pi_h \boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{L^2} + (1/\sqrt{\alpha}) \|u - Pu\|_U \right)
\end{aligned}
$$

Proof

$P = L^2$-projection onto U_h, $\Pi_h : \boldsymbol{\Sigma}^+ \to \boldsymbol{\Sigma}_h$ Fortin operator

$$
\begin{aligned}
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&= (\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}, \Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) - (\operatorname{div}(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), u - Pu)\\
&\le \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{L^2}\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2} + \|\operatorname{div}(\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{L^2}\|u - Pu\|_U\\
&\le \|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}\left(\|\Pi_h\boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{L^2} + (1/\sqrt{\alpha})\|u - Pu\|_U\right)\\
\|Pu - u_h\|_U &\le C \sup_{\boldsymbol{\tau}_h} \frac{(Pu - u_h, \operatorname{div}\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_\Sigma}\\
&\le C \sup_{\boldsymbol{\tau}_h} \frac{(Pu - u, \operatorname{div}\boldsymbol{\tau}_h) + (u - u_h, \operatorname{div}\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_\Sigma}\\
&\le C\left(\|Pu - u\|_U + \sup_{\boldsymbol{\tau}_h} \frac{-(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_\Sigma}\right)\\
&\le C\left(\|Pu - u\|_U + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{L^2}\right)
\end{aligned}
$$

Fortid condition

Definition

The spaces $\boldsymbol{\Sigma}_h$, U_h satisfy the **Fortid** condition if there exists a Fortin operator which converges strongly to the identity operator, namely
$\Pi_h : \boldsymbol{\Sigma}^+ \to \boldsymbol{\Sigma}_h$ s.t.

$$(\mathrm{div}(\sigma - \Pi_h \sigma), v) = 0 \quad \forall v \in U_h$$
$$||\Pi_h \sigma||_\Sigma \leq C ||\sigma||_{\Sigma^+}$$

$$\boxed{||I - \Pi_h||_{\mathcal{L}(\Sigma^+, L^2)} \to 0}$$

Final convergence result

Theorem

Assume ellipticity in the kernel and Fortid condition
For any $N \in \mathbb{N}$ define $\rho_N(h) :]0,1] \to \mathbb{R}$ as

$$\rho_N(h) = \sup_{u \in \bigoplus_{i=1}^{m(N)} E_i} \left(\inf_{v_h} \|u - v_h\|_U + \|\nabla u - \Pi_h \nabla u\|_{L^2} \right)$$

Then $\|T_U - T_{U,h}\|_{\mathcal{L}(U)} \to 0$ and the following estimates hold true

$$\sum_{i=1}^{m(N)} |\lambda_i - \lambda_{i,h}| \le C(\rho_N(h))^2$$

$$\hat{\delta}\left(\bigoplus_{i=1}^{m(N)} E_i, \bigoplus_{i=1}^{m(N)} E_{i,h} \right) \le C\rho_N(h)$$

Back to the counterexample

Crisscross mesh

$\boldsymbol{\Sigma}_h = \{$continuous p.w. linears (componentwise)$\}$

$U_h = \operatorname{div} \boldsymbol{\Sigma}_h \subset \{$p.w. constants$\}$

Theorem

With the above choice of spaces, there exist a sequence $\{g_h\} \subset U$ with $\|g_h\|_0 = 1$ s.t.

$$\boxed{\|u - u_h\|_U \not\to 0}$$

that is $\|T_U - T_{U,h}\|_{\mathcal{L}(U)} \not\to 0$

Back to the counterexample

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$$\boxed{||u - u_h||_U \not\to 0}$$

that is $\|T_U - T_{U,h}\|_{\mathcal{L}(U)} \not\to 0$

Proof

Boffi-Brezzi-G. 2000

Use inequality by Qin 1994 based on idea of Boland–Nicolaides 1985

A positive example

General mesh (triangles, parallelograms, tetrahedrons, parallelepipeds)

$\mathbf{\Sigma}_h$: Raviart–Thomas space of order k

U_h : $\mathcal{P}_{k-1}$ or tensor product polynomials $\mathcal{Q}_{k-1}$

A positive example

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Convergence: $O(h^{2k})$ eigenvalues, $O(h^k)$ eigenfunctions.

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Outline

Two types of eigenvalue problems

Problem of type $\binom{\mathbf{0}}{\mathbf{g}}$
Find $(\lambda, u) \in \mathbb{R} \times U$ with $u \neq 0$ such that for some $\sigma \in \Sigma$

$$\begin{cases} a(\sigma, \tau) + b(\tau, u) = 0 & \forall \tau \in \Sigma \\ b(\sigma, v) = -\lambda (u, v)_{H_U} & \forall v \in U \end{cases}$$

Laplace, biharmonic, Maxwell (2), ...

Problem of type $\binom{\mathbf{f}}{\mathbf{0}}$
Find $(\lambda, u) \in \mathbb{R} \times \Sigma$ with $u \neq 0$ such that for some $p \in Q$

$$\begin{cases} a(u, v) + b(v, p) = \lambda (u, v)_{H_\Sigma} & \forall v \in \Sigma \\ b(u, q) = 0 & \forall q \in Q \end{cases}$$

Stokes, Laplace with Lagrange multiplier, Maxwell (1)

Problem of type $\binom{\mathbf{0}}{\mathbf{g}}$

Problem of type $\binom{\mathbf{0}}{\mathbf{g}}$

Find $(\lambda, u) \in \mathbb{R} \times U$ with $u \neq 0$ such that for some $\sigma \in \Sigma$

$$\begin{cases} a(\sigma,\tau) + b(\tau,u) = 0 & \forall \tau \in \Sigma \\ b(\sigma,v) = -\lambda(u,v)_{H_U} & \forall v \in U \end{cases}$$

We set $U \subset H_U \subset U'$

Solution operator $T_U : H_U \to H_U$ such that For any $g \in H_U$ $T_U g \in U \subset H_U$ satisfies:
find $(\sigma, T_U g) \in \Sigma \times U$ such that

$$\begin{cases} a(\sigma,\tau) + b(\tau,T_U g) = 0 & \forall \tau \in \Sigma \\ b(\sigma,v) = -(g,v)_{H_U} & \forall v \in U \end{cases}$$

Abstract framework for $\binom{\mathbf{0}}{\mathbf{g}}$-Type problem

- a symmetric and positive semidefinite, and set $\|\sigma\|_a^2 = a(\sigma, \sigma)$
- We assume that the ellipticty on the kernel and the inf-sup conditions hold true for the continuous problem
- We assume that T_U is compact from H_U to U.
- $\mathbb{K}_h = \{\tau_h \in \Sigma_h : b(\tau_h, v) = 0 \ \forall v \in U_h\}$

Weak approximability of U^+ [WAg]
There exists $\omega_1(h) \to 0$ as $h \to 0$ such that

$$b(\sigma_h, v) \leq \omega_1(h)\|v\|_{U^+}\|\sigma_h\|_a \quad \forall \sigma_h \in \mathbb{K}_h \ \forall v \in U^+$$

Strong approximability of U^+ [SAg]
There exists $\omega_2(h) \to 0$ as $h \to 0$ such that for every $v \in U^+$ there exists $v^I \in \mathbb{K}_h$ such that

$$\|v - v^I\|_0 \leq \omega_2(h)\|v\|_{U^+}$$

Abstract framework (cont'd)

Fortin operator

$\Pi_h : \Sigma^+ \to \Sigma_h$ such that for all $\sigma \in \Sigma^+$

$$\begin{cases} b(\sigma - \Pi_h\sigma, v_h) = 0 & \forall v_h \in U_h \\ \|\Pi_h\sigma\|_\Sigma \le C\|\sigma\|_{\Sigma^+} \end{cases}$$

Fortid property [FORTID]

There exists $\omega_3(h) \to 0$ as $h \to 0$ such that

$$\|\sigma - \Pi_h\sigma\|_a \le \omega_3(h)\|\sigma\|_{\Sigma^+}$$

Theorem

[WAg], [SAg] and [FORTID] imply that $T_{U,h}$ converges to T_U uniformly from H_U into U, that is there exists $\omega_g(h) \to 0$ as $h \to 0$ such that

$$\|T_{U,h}g - T_U g\|_U \le \omega_g(h)\|g\|_{H_U} \quad \text{for all } g \in H_U.$$

Moreover, $\quad \|T_{U,h} - T_U\|_{\mathcal{L}(H_U)} \to 0.$

Problem of type $\binom{\mathbf{f}}{\mathbf{0}}$

Problem of type $\binom{\mathbf{f}}{\mathbf{0}}$
Find $(\lambda, u) \in \mathbb{R} \times V$ with $u \neq 0$ such that for some $p \in Q$

$$\begin{cases} a(u,v) + b(v,p) = \lambda(u,v)_{H_V} & \forall v \in V \\ b(u,q) = 0 & \forall q \in Q \end{cases}$$

We set: $V \subset H_V \subset V'$
Solution operator $T_V : H_V \to H_V$ such that For any $f \in H_V$ $T_V f \in V \subset H_V$ satisfies:
Find $(T_V f, p) \in V \times Q$ such that

$$\begin{cases} a(T_V f, v) + b(v,p) = (f,v)_{H_V} & \forall v \in V \\ b(u,q) = 0 & \forall q \in Q \end{cases}$$

Abstract framework for $\binom{\mathbf{f}}{\mathbf{0}}$-Type problem

- We assume that the ellipticity in the kernel and the inf-sup conditions are satisfied for the continuous problem
- We assume that T_V is compact from H_V to V.
- $\mathbb{K}_h = \{v_h \in V_h : b(v_h, q) = 0 \ \forall q \in Q_h\}$

Ellipticity in the discrete kernel [ELKER]
There exists $\alpha > 0$ such that $a(v_h, v_h) \geq \alpha \|v_h\|_V \quad \forall v_h \in \mathbb{K}_h$.

Weak approximability of Q^+ [WAf]
There exists $\omega_4(h) \to 0$ as $h \to 0$ such that for all $q \in Q^+$

$$\sup_{v_h \in \mathbb{K}_h} \frac{b(v_h, q)}{\|v_h\|_V} \leq \omega_4(h) \|q\|_{Q^+}$$

Strong approximability of V^+ [SAf]
There exists $\omega_5(h) \to 0$ as $h \to 0$ such that for every $v \in V^+$ there exists $v^I \in \mathbb{K}_h$ such that

$$\|v - v^I\|_V \leq \omega_5(h) \|v\|_{V^+}$$

Abstract framework (cont'd)

Theorem

[ELKER], [WAf], [SAf] imply that $T_{V,h}$ converge uniformly to T_V in $\mathcal{L}(H_V, V)$, that is there exists $\omega_f(h) \to 0$ as $h \to 0$ such that

$$\|T_{V,h}f - T_V f\|_V \leq \omega_f(h)\|f\|_{H_V}$$