Finite Element Approximation of Eigenvalue Problems
Lesson 2

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Buenos Aires, 28 July 2026

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Outline

Spectral theory for compact operators

Variationally posed eigenproblem

Babuška-Osborn theory

Non standard Galerkin approximation

Fundamental result

Kato 1995

Let X be an Hilbert space and let $T : X \to X$ be a compact linear self-adjoint operator.
We consider a family of compact operators $T_h : X \to X$ such that

$$\|T - T_h\|_{\mathcal{L}(X)} \to 0 \quad \text{as } h \to 0$$

Theorem

For any compact set $K \subset \rho(T)$, there exists $h_0 > 0$ such that, for all $h < h_0$, we have $K \subset \rho(T_h)$ (absence of spurious modes). If λ is a nonzero eigenvalue of T with algebraic multiplicity equal to m, then there are m eigenvalues $\lambda_h^{(1)}, \dots, \lambda_h^{(m)}$ of T_h, repeated according to their algebraic multiplicities, such that each $\lambda_h^{(i)}$ converges to λ as $h \to 0$.
Moreover, the gap between the direct sum of the generalized eigenspaces associated with $\lambda_h^{(1)}, \dots, \lambda_h^{(m)}$ and the generalized eigenspace associated to λ tends to zero as $h \to 0$.

Gap between Hilbert spaces

Definition of gap

Given two closed subspaces E, F of H, we define

$$\delta(E,F) = \sup_{\substack{u\in E\\ \|u\|_H=1}} \inf_{v\in F} \|u-v\|_H, \quad \hat{\delta}(E,F) = \max(\delta(E,F),\delta(F,E)).$$

$\hat{\delta}(E,F)$ is called the gap between E and F.

Convergence of eiegnvalues and eigenfunctions

Some notation

Let $m : \mathbb{N} \to \mathbb{N}$ such that for $N \in \mathbb{N}$ $\lambda^{(m(1))} < \lambda^{(m(2))} < \cdots \lambda^{(m(N))}$. Clearly, $\lambda^{(m(1))} < \lambda^{(m(2))} < \cdots \lambda^{(m(N))}$ are the first N distinct eigenvalues, and $m(N)$ is the dimension of the space generated by the eigenspaces of the first N distinct eigenvalues.

Definition of convergence

$\forall \varepsilon > 0$, $\forall N \in \mathbb{N}$, $\exists h_0 > 0$ such that $\forall h \leq h_0$

$$\max_{i=1,\dots,m(N)} |\lambda^{(i)} - \lambda_h^{(i)}| \leq \varepsilon$$

$$\hat{\delta}\left(\bigoplus_{i=1}^{m(N)} E^{(i)}, \bigoplus_{i=1}^{m(N)} E_h^{(i)}\right) \leq \varepsilon$$

Convergence

Uniform convergence

$$\lim_{h\to 0}\|T - T_h\|_{\mathcal{L}(H)} = 0$$

Theorem

If T is self-adjoint and compact

$$\boxed{\text{Uniform convergence}} \iff \boxed{\text{Eigenmodes convergence}}$$

See: Kato 1995 for the sufficient part, Boffi-Brezzi-G 2000 for the necessity part.

Strategy

1. prove uniform convergence
2. estimate the order of convergence

Convergence

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Strategy

1. prove uniform convergence
2. estimate the order of convergence

Remark If $T : V \to V$ is compact, we can work with the uniform convergence in $\mathcal{L}(V)$ that is $\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(V)} = 0$. This would ensure an analogous eigenmodes convergence with natural modifications.

Outline

Variationally posed eigenproblem

Abstract framework H $(= L^2(\Omega))$, V $(= H^1_0(\Omega)) \subset H$
Hilbert spaces, V compactly embedded in H

$a(u,v)$ $(= (\nabla u, \nabla v))$ $\quad V \times V \to \mathbb{R}$
bilinear, continuous, symmetric, coercive

$b(u,v)$ $(= (u,v))$ $\quad H \times H \to \mathbb{R}$
bilinear, continuous, symmetric
Eigenvalue problem Find $\lambda \in \mathbb{R}$ and $u \in V$ with $u \neq 0$, such that

$$\text{(EIG)} \quad a(u,v) = \lambda b(u,v) \quad \forall v \in V.$$

Solution operator

Solution operator T

Given $f \in H$, $Tf \in V$ is the unique solution of the **source problem**

$$a(Tf, v) = b(f, v) \quad \forall v \in V.$$

NB: $T(H) \subset V$ implies T is a compact operator.

$0 < \lambda^{(1)} \le \lambda^{(2)} \le \cdots \le \lambda^{(k)} \le \cdots$

$E^{(k)} = \mathrm{span}(u^{(k)})$, normalization $b(u^{(k)}, u^{(k)}) = 1$, $V = \bigoplus_{i=1}^{\infty} E^{(i)}$

Approximation

$V_h \subset V$, $\dim V_h = N$

Discrete eigenvalue problem

Find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$ with $u_h \neq 0$, such that

$$(\mathrm{EIG}_h) \quad a(u_h, v) = \lambda_h b(u_h, v) \quad \forall v \in V_h.$$

Discrete solution operator T_h

Given $f \in H$, $T_h f \in V_h$ is the unique solution of

$$a(T_h f, v) = b(f, v) \quad \forall v \in V_h.$$

$0 < \lambda_h^{(1)} \leq \lambda_h^{(2)} \leq \cdots \leq \lambda_h^{(N)}$

$E_h^{(k)} = \mathrm{span}(u_h^{(k)})$, normalization $b(u_h^{(k)}, u_h^{(k)}) = 1$, $V_h = \bigoplus_{i=1}^{N} E_h^{(i)}$

Convergence

Uniform convergence

$$\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(H)} = 0$$

Theorem

If T is self-adjoint and compact

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Strategy

1. prove uniform convergence
2. estimate the order of convergence

Galerkin approximation of compact operators

Bramble-Osborn 1973
Osborn 1975
Kolata 1978

Elliptic projection Let $\Pi_h : V \to V_h$ the elliptic projection associated to the bilinear for a, that is: for all $u \in V$, $\Pi_h u \in V_h$ such that

$$a(u - \Pi_h u, v) = 0 \quad \forall v \in V_h.$$

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$$a(u - \Pi_h u, v) = 0 \quad \forall v \in V_h.$$

$$\boxed{T_h = \Pi_h T}$$

We have

$$a(Tf - \Pi_h Tf, v) = 0 \quad \forall v \in V_h, \quad a(Tf - T_h f, v) = 0 \quad \forall v \in V_h$$

definition of Π_h Galerkin orthogonality

By uniqueness we conclude: $T_h f = \Pi_h Tf$ or $(I - \Pi_h)Tf = 0.$

An abstract convergence result

Theorem

If T is compact from H to V and Π_h converges strongly to the identity operator from V to H, then the uniform convergence

$$\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(H)} = 0$$

holds true.

Remark The compactness required in this theorem is stronger than that we assumed in general. However, for the case of eigenvalue problem associated to elliptic differential operator, it can be shown thanks to the regularity of the solution of the source problem.

Remark If T is compact from V in V, using a stronger pointwise convergence of Π_h to the identity, from V into V, we can use the same proof to show the uniform convergence

$$\lim_{h\to 0} \|T - T_h\|_{\mathcal{L}(V)} = 0$$

Proof

$\boxed{\text{Step 1}}$ We show $\|I - \Pi_h\|_{\mathcal{L}(V,H)} \le C$.
For any $u \in V$, define $c(h, u)$ by $\|(I - \Pi_h)u\|_H = c(h, u)\|u\|_V$.
Strong (pointwise) convergence implies that $c(h, u) \to 0$ for each u.

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Uniform boundedness principle (or Banach–Steinhaus thm) implies that there exists C s.t. $\|I - \Pi_h\|_{\mathcal{L}(V,H)} \le C$.

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$\boxed{\text{Step 2}}$ We show $\|T - T_h\|_{\mathcal{L}(H)} \to 0$
Let $\{f_h\} \in H$, such that $\|f_h\|_H = 1$ and $\|T - T_h\|_{\mathcal{L}(H)} = \|(T - T_h)f_h\|_H$.

Proof

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We have $T_h v = \Pi_h T v = \Pi_h w$, so $T_h v \to w$.
For any $\varepsilon > 0$ there exists h small enough such that: $\|Tf_h - w\|_V + \|(I - \Pi_h)w\|_H \le \varepsilon$.

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For any $\varepsilon > 0$ there exists h small enough such that: $\|Tf_h - w\|_V + \|(I - \Pi_h)w\|_H \le \varepsilon$.
Hence $\|(I - \Pi_h)Tf_h\|_H \le \|(I - \Pi_h)(Tf_h - w)\|_H + \|(I - \Pi_h)w\|_H \le (C + 1)\varepsilon$.

Conclusion on standard Galerkin approximation

$$\boxed{\|T - T_h\|_{\mathcal{L}(H)} \to 0}$$

Any finite element choice which provides a (pointwise) convergent scheme for the approximation of a problem with compact resolvent can be successfully applied to the approximation of the corresponding eigenvalue problem (uniform convergence).

Direct proof can be obtained by using a priori error estimates Let $u \in V$ and $u_h \in V_h$ be the continuous and discrete solutions of the associated source problem corresponding to the datum $f \in H$:

$$\|(T - T_h)f\|_H = \|u - u_h\|_H \leq Ch^\alpha \|f\|_H$$

for some $\alpha > 0$

Outline

Approximation of eigenvectors

Theorem

Let μ be a non-zero eigenvalue of T, let $E = E(\mu)X$ be its generalized eigenspace, and let $E_h = E_h(\mu)X$. Then

$$\hat{\delta}(E, E_h) \le C \|(T - T_h)|_E\|_{\mathcal{L}(X)}.$$

Corollary

Let λ be an eigenvalue of (EIG), let $E = E(\lambda^{-1})V$ be its generalized eigenspace, and let $E_h = E_h(\lambda^{-1})V$. Then

$$\hat{\delta}(E, E_h) \le C \sup_{\substack{u \in E \\ \|u\|_V = 1}} inf_{v \in V_h} \|u - v\|_V.$$

Approximation of eigenvalues

Theorem

Let μ be a non-zero eigenvalue of T with algebraic multiplicity m and let $\hat{\mu}_h$ denote the arithmetic mean of the m discrete eigenvalues of T_h converging towards μ. Let $\phi_1, \dots, \phi_m$ be a basis of generalized eigenvectors in $E = E(\mu)X$. Then

$$|\mu - \hat{\mu}_h| \le \frac{1}{m} \sum_{i,j=1}^{m} |((T - T_h)\phi_i, \phi_j)| + C\|(T - T_h)|_E\|^2_{\mathcal{L}(X)}.$$

Corollary

Let λ be an eiegenvalue of (EIG) and let $\hat{\lambda}_h$ denote the arithmetic mean of the m discrete eigenvalues of (EIG$_h$) converging toward λ. Then

$$|\lambda - \hat{\lambda}_h| \le C\Big(\sup_{\substack{u\in E\\ \|u\|_V=1}} inf_{v\in V_h}\|u - v\|_V\Big)^2$$

Application to the Laplace eigenvalue problem

Given $\Omega \subset \mathbb{R}^d$, $H = L^2(\Omega)$, $V = H^1_0(\Omega)$;
$a(u,v) = (\nabla u, \nabla v)$, $\quad b(u,v) = (u,v)$

Laplace eigenvalue problem

Find $\lambda \in \mathbb{R}$ and $u \in V$ with $u \neq 0$, such that

$$\text{(EIG)} \quad a(u,v) = \lambda b(u,v) \quad \forall v \in V.$$

Discrete Laplace eigenvalue problem

Find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$ with $u_h \neq 0$, such that

$$(\text{EIG}_h) \quad a(u_h,v) = \lambda_h(u_h,v) \quad \forall v \in V_h.$$

Solution operator

We can introduce $T : V \to V$ such that for $f \in V$

$$a(Tf, v) = (f, v) \quad \forall v \in V$$

If $f \in H$, the above equation has a unique solution $Tf \in V \subset H$.

Hence we are led to two admissible definitions: $T_V : V \to V$ and $T_H : H \to H$.

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Proposition

- If V is endowed with the norm induced by the bilinear form a, then the operator T_V is self-adjoint.
- The operator T_H is self-adjoint.

$\boxed{T_V \text{ case}}$

$$a(T_V x, y) = (x, y) = (y, x) = a(T_V y, x) = a(x, T_V y) \quad \forall x, y \in V.$$

$\boxed{T_H \text{ case}}$

$$(T_H x, y) = (y, T_H x) = a(T_H y, T_H x) = a(T_H x, T_H y) = (x, T_H y) \quad \forall x, y \in H.$$

Discrete Laplace eigenvalue problem

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Discrete Laplace eigenvalue problem

Find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$ with $u_h \neq 0$, such that

$$(\text{EIG}_h) \quad a(u_h, v) = \lambda_h(u_h, v) \quad \forall v \in V_h.$$

Approximation property of the finite element space

$$\inf_{v \in V_h} \|u - v\|_{L^2(\Omega)} \leq C h^{\min(k+1,r)} \|u\|_{H^r(\Omega)}$$

$$\inf_{v \in V_h} \|u - v\|_{H^1_0(\Omega)} \leq C h^{\min(k,r-1)} \|u\|_{H^r(\Omega)}$$

Analysis for $T = T_V$

We define $T_h : V \to V$ such that for $T_h f \in V_h \subset V$

$$a(T_h f, v) = (f, v) \quad \forall v \in V_h$$

To show convergence we can use either the Kolata result or direct error estimate: in the latter case one can show that for some $\varepsilon > 0$

$$\|Tf - T_h f\|_V \leq Ch^{\varepsilon} \|f\|_V$$

Estimate of the rate of convergence

We assume that the regularity of the eigenspace E associated to λ is r, that is $E \subset H^r(\Omega)$. Hence

$$\hat{\delta}(E, E_h) \leq C\|(T - T_h)|_E\|_{\mathcal{L}(V)} = O(h^{\min(k,r-1)})$$

Error estimate for eigenfunctions

Let u be a unit eigenfunction associated with the eigenvalue λ of multiplicity m, and let $w_h^{(1)}, \dots, w_h^{(m)}$ denote eigenfunctions associated with the m discrete eigenvalues converging to λ. Then there exists

$$u_h \in \mathrm{span}(w_h^{(1)}, \dots, w_h^{(m)})$$

such that

$$\|u - u_h\|_V \leq Ch^\tau \|u\|_{H^{1+\tau}(\Omega)}$$

where $\tau = \min(k, r - 1)$.

Rate of convergence for eigenvalues

Theorem

Let λ_h be an eigenvalue converging to λ. Then the following optimal double order of convergence holds:

$$\lambda \leq \lambda_h \leq \lambda + Ch^{2\tau}.$$

We apply the result

$$|\lambda - \lambda_h| \leq \sum_{i,j=1}^{m} |((T - T_h)\phi_i, \phi_j)| + C\|(T - T_h)|_E\|^2_{\mathcal{L}(X)}.$$

We need to bound $\sum_{i,j=1}^{m} |((T - T_h)\phi_i, \phi_j)|$.
We have for $u, v \in E$

$$|((T - T_h)u, v)_V| = |a((T - T_h)u, v)| = C \inf_{v_h \in V_h} |a((T - T_h)u, v - v_h)|$$
$$\leq \|(T - T_h)u\|_V \inf_{v_h \in V_h} \|v - v_h\|_V \leq Ch^\tau \|u\|_V h^\tau \|v\|_{H^{1+\tau}(\Omega)} \leq Ch^{2\tau} \|u\|_V \|v\|_V$$

Some remarks on the rate of convergence for eigenvalues

Knyazev 1986

Using the *min-max* representation of eigenvalues, for $i = 1, \dots, N$ we have

$$0 \le \frac{\lambda_h^{(i)} - \lambda^{(i)}}{\lambda_h^{(i)}} \le \delta^2(E_{1,\dots,i}, V_h) = \|(I - \Pi_h)P_{1,\dots,i}\|^2_{\mathcal{L}(V)}$$

where $P_{1,\dots,i}$ is the elliptic projection onto $E_{1,\dots,i} \subset V$ the span of the first i eiegenvectors $u^{(1)}, \dots, u^{(i)}$.

Moreover, let $\lambda^{(p)}$ be an eigenvalue of multiplicity m, that is

$$\lambda^{(p-1)} < \lambda^{(p)} = \dots = \lambda^{(p+m-1)} < \lambda^{(p+m)} \quad \text{with } p + m - 1 \le N.$$

Then for any index $i = p, \dots, p + m - 1$ we have

$$0 \le \frac{\lambda_h^{(i)} - \lambda^{(p)}}{\lambda_h^{(i)}} \le \inf_{\substack{E_{1,\dots,p-1} \subset E_{1,\dots,i} \subseteq E_{1,\dots,p+m-1} \\ \dim E_{1,\dots,i} = i}} \delta^2(E_{1,\dots,i}, V_h) = \delta^2(E_{1,\dots,p+m-1}, V_h).$$

Improved error estimate

Knyazev-Osborn 2006

Let us fix an index i with $1 \le i \le N$ such that

$$\min_{j=1,\dots,i-1} |\lambda_h^{(j)} - \lambda^{(i)}| \neq 0,$$

then

$$0 \le \frac{\lambda_h^{(i)} - \lambda^{(i)}}{\lambda_h^{(i)}} \le \frac{\|(I - P_h + P_{1,\dots,i-1,h})u^{(i)}\|_V^2}{\|u^{(i)}\|_V^2}$$

$$\le \left(1 + \max_{j=1,\dots,i-1} \frac{(\lambda_h^{(j)}\lambda^{(j)})^2}{|\lambda_h^{(j)} - \lambda^{(i)}|^2} \|(I - P_h)TP_{1,\dots,i-1,h}\|_{\mathcal{L}(V)}^2\right) \delta^2(u^{(i)}, V_h),$$

where $P_{1,\dots,i-1,h}$ is the elliptic projection onto $E_{1,\dots,i-1,h} = \mathrm{span}\{u_h^{(1)}, \dots, u_h^{(i-1)}\}$ defined as follows: for $u \in V$, $P_{1,\dots,i-1,h}u \in E_{1,\dots,i-1,h}$ such that

$$a(u - P_{1,\dots,i-1,h}u, v) = 0 \qquad \forall v \in E_{1,\dots,i-1,h},$$

Outline

Spectral theory for compact operators

Variationally posed eigenproblem

Babuška-Osborn theory

Non standard Galerkin approximation

Variationally posed eigenvalue problem

Abstract setting

Let V and H two Hilbert spaces with $V \subset H$.
Let $a : V \times V \to \mathbb{R}$ and $b : H \times H \to \mathbb{R}$ symmetric and continuous such that

- there exists a positive constant α such that $a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in V$
- $b(u, u) > 0 \quad \forall u \in H,$ with $u \neq 0$

Problem Find $\lambda \in \mathbb{R}$ such that there exists $u \in V$ with $u \neq 0$ satisfying

$$a(u, v) = \lambda b(u, v) \quad \forall v \in V$$

Solution operator $T : V \to V$ s.t. for all $f \in V$, Tf is the solution of the *source problem*

$$a(Tf, v) = b(f, v) \quad \forall v \in V$$

Non standard approximation of the eigenvalue problem

Boffi-Gardini-G. 2026

Let V_h be a finite dimensional subspace of V.
Let $a_h : V_h \times V_h \to \mathbb{R}$ and $b_h : V_h \times V_h \to \mathbb{R}$ continuous with

$$a_h(v_h, v_h) \geq \underline{\alpha} \|v_h\|_V^2 \quad \forall v_h \in V_h, \qquad b_h(v_h, v_h) \geq 0 \quad \forall v_h \in V_h$$

Discrete problem Find $\lambda_h \in \mathbb{R}$ such that there exists $u_h \in V_h$ with $u_h \neq 0$ satisfying

$$a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h) \quad \forall v_h \in V_h$$

Generalized algebraic eigenvalue system

$$\mathsf{A}\mathsf{x} = \lambda \mathsf{B}\mathsf{x}$$

NB

- If $b_h(v_h, v_h) > 0$ then we have $N_h := \dim(V_h)$ positive discrete eigenvalues.
- If $b_h(v_h, v_h) \geq 0$ the matrix B might not be full rank. In this case $\lambda = \infty$ is an eigenvalue with multiplicity equal to the dimension of $\ker(\mathsf{B})$.

Discrete solution operator when $b_h(v_h, v_h) \geq 0$

Discrete source problem Given $f \in H$, find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = b_h(f, v_h) \quad \forall v_h \in V_h$$

Thanks to the coercivity of a, there exists a unique solution for any f.

Discrete solution operator $\widetilde{T}_h : V_h \to V_h$ s.t. for all $f \in V_h$, $\widetilde{T}_h f$ is the solution of

$$a_h(\widetilde{T}_h f, v_h) = b_h(f, v_h) \quad \forall v_h \in V_h$$

Discrete solution operator when $b_h(v_h, v_h) \geq 0$

Discrete source problem Given $f \in H$, find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = b_h(f, v_h) \quad \forall v_h \in V_h$$

Thanks to the coercivity of a, there exists a unique solution for any f.

Discrete solution operator $\widetilde{T}_h : V_h \to V_h$ s.t. for all $f \in V_h$, $\widetilde{T}_h f$ is the solution of

$$a_h(\widetilde{T}_h f, v_h) = b_h(f, v_h) \quad \forall v_h \in V_h$$

But if $b_h(f, v_h) = 0$ for all $v_h \in V_h$, $\widetilde{T}_h f$ vanishes.

Spectrum of $\widetilde{T}_h$

$\widetilde{\mu}_h \in \mathbb{R}$ is an eigenvalue of $\widetilde{T}_h$ if there exists $\widetilde{u}_h \in V_h$ with $\widetilde{u}_h \neq 0$ such that

$$\widetilde{T}_h \widetilde{u}_h = \widetilde{\mu}_h \widetilde{u}_h$$

- $\widetilde{T}_h$ has exactly N_h eigenvalues
- $\widetilde{\mu}_h = 0$ with eigenspace $\mathbb{K}_{b,h} = \{w_h \in V_h : b_h(w_h, v_h) = 0 \quad \forall v_h \in V_h\}$
- the remaining eigenvalues are positive with $\widetilde{u}_h$ not belonging to $\mathbb{K}_{b,h}$

Spectrum of $\widetilde{T}_h$

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- the remaining eigenvalues are positive with $\widetilde{u}_h$ not belonging to $\mathbb{K}_{b,h}$
- for $\widetilde{\mu}_h > 0$, $(\lambda_h, \widetilde{u}_h)$ with $\lambda_h = \dfrac{1}{\widetilde{\mu}_h}$ is an eigensolution of

$$a_h(\widetilde{u}_h, v_h) = \lambda_h b_h(\widetilde{u}_h, v_h) \quad \forall v_h \in V_h$$

Spectrum of $\widetilde{T}_h$

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$$a_h(\widetilde{u}_h, v_h) = \lambda_h b_h(\widetilde{u}_h, v_h) \quad \forall v_h \in V_h$$

- If $v_h \in \mathbb{K}_{b,h}$, then $a_h(\widetilde{u}_h, v_h) = 0$, so that $\widetilde{u}_h$ belongs to

$$\mathbb{K}_{b,h}^{\perp} = \{w_h \in V_h : a_h(w_h, v_h) = 0 \quad \forall v_h \in \mathbb{K}_{b,h}\}.$$

Discrete eigenvalue problem

Problem Find $\lambda_h \in \mathbb{R}$ such that there exists $u_h \in \mathbb{K}_{b,h}^{\perp}$ with $u_h \neq 0$ satisfying

$$a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h) \quad \forall v_h \in \mathbb{K}_{b,h}^{\perp}$$

Let $T_h : \mathbb{K}_{b,h}^{\perp} \to \mathbb{K}_{b,h}^{\perp}$ be the restriction of $\widetilde{T}_h$ to $\mathbb{K}_{b,h}^{\perp}$.

Using Descloux-Nassif-Rappaz '78, we obtain **convergence** of the discrete eigenmodes if we show:

P1 $\displaystyle \sup_{f_h \in \mathbb{K}_{b,h}^{\perp}} \frac{\|(T - T_h)f_h\|_V}{\|f_h\|_V}$ tends to 0 as h goes to 0

P2 $\displaystyle \inf_{v_h \in \mathbb{K}_{b,h}^{\perp}} \|u - v_h\|_V$ tends to 0 as h goes to 0 for all $u \in V_0$

Virtual Element Method

Beirão da Veiga-Brezzi-Cangiani-Manzini-Marini-Russo '13
Ahmad-Alsaedi-Brezzi-Marini-Russo '13

$\mathcal{T}_h$ family of polygonal decomposition of $\Omega \subseteq \mathbb{R}^2$.

Local VEM space

$$\begin{aligned} V_h^k(P) =&\{v_h \in H^1(P) : v_h \in C^0(\partial P),\ v_h|_e \in \mathcal{P}^k(e)\ \forall e \in \partial P, \\ &\Delta v_h \in \mathcal{P}^k(P),\ (\Pi_k^\nabla v_h - v_h, q)_P = 0\ \forall q \in \mathcal{M}^{k,k-1}(P)\}\end{aligned}$$

Computable projection operators

$\Pi_k^\nabla : V_h^k(P) \to \mathcal{P}_k(P)$ H^1-projection operator

$$a^P(\Pi_k^\nabla v_h - v_h, p) = (\nabla(\Pi_k^\nabla v_h - v_h), \nabla p)_P = 0 \quad \forall p \in \mathcal{P}_k(P),$$

$$(\Pi_k^\nabla v_h - v_h, 1)_P = 0, \quad \text{for } k \geq 2, \qquad (\Pi_k^\nabla v_h - v_{,h} 1)_{\partial P} = 0, \quad \text{for } k = 1$$

$\Pi_k^0 : V_h^k(P) \to \mathcal{P}_k(P)$ L^2-projection operator

$$b^P(\Pi_k^0 v_h - v_h, p) = (\Pi_k^0 v_h - v_h, p)_P = 0 \quad \forall p \in V_h^k(P)$$

VEM approximation

Local discrete bilinear forms

$$a_h^P(u_h, v_h) = a^P(\Pi_k^\nabla u_h, \Pi_k^\nabla v_h) + S_a^P((I - \Pi_k^\nabla)u_h, (I - \Pi_k^\nabla)v_h)$$

$$b_h^P(u_h, v_h) = b^P(\Pi_k^0 u_h, \Pi_k^0 v_h) + S_b^P((I - \Pi_k^0)u_h, (I - \Pi_k^0)v_h),$$

with S_a^P and S_b^P symmetric positive definite bilinear forms such that

$$c_0 a^P(v_h, v_h) \le S_a^P(v_h, v_h) \le c_1 a^P(v_h, v_h) \qquad \forall v_h \in V_h^k(P) \text{ with } \Pi_k^\nabla v_h = 0$$

$$c_2 b^P(v_h, v_h) \le S_b^P(v_h, v_h) \le c_3 b^P(v_h, v_h) \qquad \forall v_h \in V_h^k(P) \text{ with } \Pi_k^0 v_h = 0,$$

Matrix form

$$\mathsf{A}^P = \mathsf{A}_1^P + \alpha_P \mathsf{A}_2^P, \qquad \mathsf{B}^P = \mathsf{B}_1^P + \beta_P \mathsf{B}_2^P$$

VEM approximation

Gardini-Vacca '18
Gardini-Manzini-Vacca '19

Global VEM space $V_h = \{v \in H^1_0(\Omega) : v|_P \in V_h^k(P)\ \forall P \in \mathcal{T}_h\}$

Global discrete bilinear forms

$$a_h(u_h, v_h) = \sum_{P \in \mathcal{T}_h} a_h^P(u_h, v_h), \qquad b_h(u_h, v_h) = \sum_{P \in \mathcal{T}_h} b_h^P(u_h, v_h)$$

Discrete eigenvalue problem

Find $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ with $u_h \neq 0$ such that $a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h) \quad \forall v_h \in V_h$

VEM approximation

Gardini-Vacca '18
Gardini-Manzini-Vacca '19

Global VEM space $V_h = \{v \in H_0^1(\Omega) : v|_P \in V_h^k(P)\ \forall P \in \mathcal{T}_h\}$

Global discrete bilinear forms

$$a_h(u_h, v_h) = \sum_{P \in \mathcal{T}_h} a_h^P(u_h, v_h), \qquad b_h(u_h, v_h) = \sum_{P \in \mathcal{T}_h} b_h^P(u_h, v_h)$$

Discrete eigenvalue problem

Find $(\lambda_h, u_h) \in \mathbb{R} \times V_h$ with $u_h \neq 0$ such that $a_h(u_h, v_h) = \lambda_h b_h(u_h, v_h) \quad \forall v_h \in V_h$

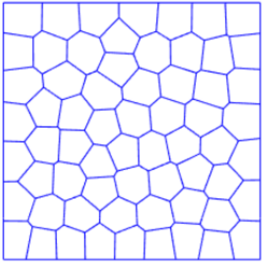
Discrete solution operator $T_h : H \to V_h \subset H$:
given $f \in H$ $T_h f \in V_h$ such that $a_h(T_h f, v_h) = b_h(f, v_h)\ \forall v_h \in V_h$

Theorem Gardini-Vacca '18 $\|T - T_h\|_{\mathcal{L}(H,H)} \to 0$ for all fixed values $\alpha, \beta \geq 0$.

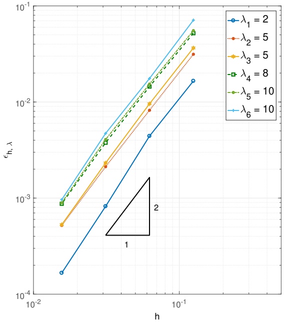
Virtual element results - convergence

Matlab code based on Virtual Element @ Bicocca

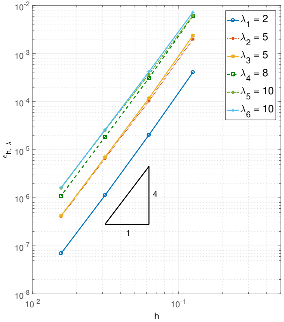
Fixed parameters α_P and β_P mean value of the eigenvalues of A_1^P and B_1^P



Voronoi mesh



$k = 1$

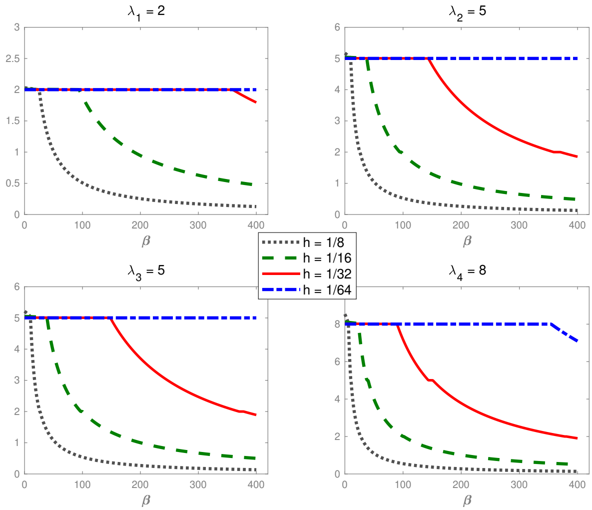


$k = 2$

Virtual element results - effect of parameters

$\alpha_P = \alpha = 10$,

$\beta_P = \beta$
with $0 \leq \beta \leq 400$



Convergence of VEM without stabilizing the right hand side

Proposition 1

P1 holds true for any choice of the VEM space.
There exists $\omega(h)$ tending to 0 as $h \to 0$, such that for all $f_h \in \mathbb{K}_{b,h}^{\perp}$ it holds

$$\|(T - T_h)f_h\|_{1,\Omega} \leq \omega(h)\|f_h\|_{1,\Omega}.$$

Proposition 2

Let us consider $k = 1$ and $k = 2$, then for any choice of the mesh, property **P2** holds true.

The proof is based on the fact that $\Pi_k^{\nabla} v_h = \Pi_k^0 v_h$ for all $v_h \in V_h$.

Dimension of kernel of B

Triangular mesh

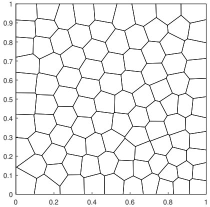
$\mathcal{T}_h$	$\dim(\mathbb{K}_{b,h})$ ($\dim(V_h)$)		
N	$k=1$	$k=2$	$k=3$
4	0 (9)	0 (81)	0 (185)
8	0 (49)	0 (353)	0 (785)
16	0 (225)	0 (1473)	0 (3233)
32	0 (961)	0 (6017)	0 (13121)
64	0 (3969)	0 (24321)	- (52865)

Square mesh

$\mathcal{Q}_h$	$\dim(\mathbb{K}_{b,h})$ ($\dim(V_h)$)		
N	$k=1$	$k=2$	$k=3$
4	0 (9)	0 (49)	0 (105)
8	0 (49)	0 (225)	0 (465)
16	0 (225)	0 (961)	0 (1953)
32	0 (961)	0 (3969)	0 (8001)
64	0 (3969)	0 (16129)	0 (32385)

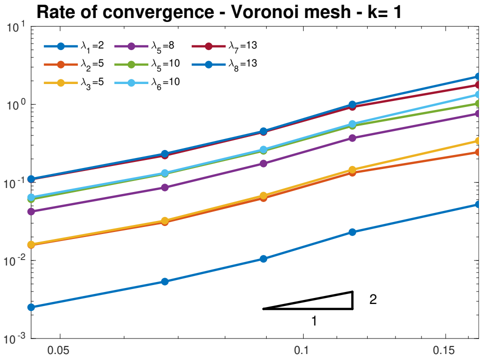
Dimension of kernel of B

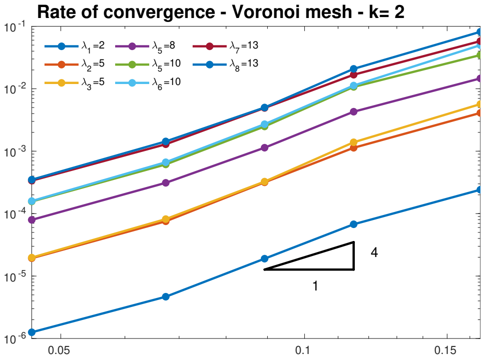
Voronoi mesh



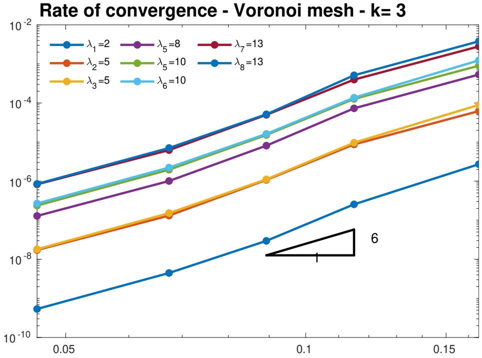
$\mathcal{V}_h$	$\dim(\mathbb{K}_{b,h})$ ($\dim(V_h)$)			
N	$k = 1$	$k = 2$	$k = 3$	$k = 4$
50	0 (73)	0 (245)	0 (467)	5 (739)
100	0 (163)	0 (525)	1 (987)	57 (1549)
200	0 (346)	0 (1091	43 (2036)	188 (3181)
400	0 (727)	0 (2253)	182 (4179)	512 (6505)
800	0 (1500)	0 (4599)	504 (8498)	1212 (13197)

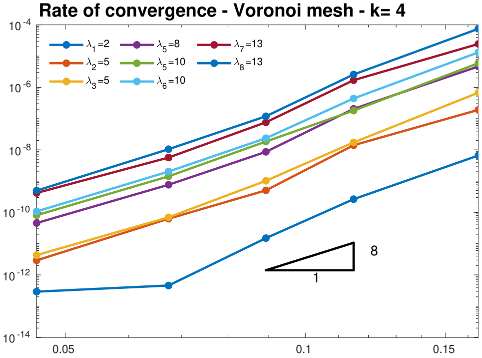
Numerical results - Voronoi mesh





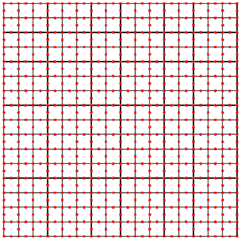
Numerical results - Voronoi mesh





Dimension of kernel of B

Dyadic mesh (courtesy of Credali-Gradini)



$\mathcal{D}_h$	$\dim(\mathbb{K}_{b,h})$ $(\dim(V_h))$	
N	$k=1$	$k=2$
8	82 (225)	305 (577)
16	290 (883)	1121 (2177)
32	1090 (3201)	4289 (8449)
64	4226 (12545)	16769 (33281)

Numerical results - Dyadic mesh