Finite Element Approximation of Eigenvalue Problems
Lesson 1

Lucia Gastaldi

Università di Brescia - Italy

Buenos Aires, 27 July 2026

UNIVERSITÀ
DEGLI STUDI
DI BRESCIA

Plan of the course

Lesson 1 The symmetric coercive case: where everything works
Lesson 2 Spectral approximation of compact operators

Babuška - Osborn 1991
D. Boffi Acta Numerica 2010

Lesson 3 Mixed formulations. Appearance of spurious eigenvalues
Lesson 4 Error control and adaptivity

Outline

Examples

One dimensional examples

Two dimensional Laplace eigenvalue problem

Variationally posed eigenvalue problem

Convergence

Eigenvalue problem

An eigenvalue problem solves the equation

$$Ax = \lambda x$$

That is, we have to find scalars λ **eigenvalues** and vectors x **eigenvectors** such that a linear transformation A only scales the vector without changing its direction.

These problems are used across physics, structural mechanics, and data analysis.

Examples

- Vibrations

Babuška- Osborn 1991

- Time harmonic Maxwell's equations
- Photonic band gaps
- Schrödinger equation
- ...

The transverse vibration of a string

We study small, transverse vibration of a homogeneous string of length L stretched between the two endpoints.
$v(x,t)$ the vertical displacement satisfies

$$
\begin{aligned}
&-p\frac{\partial^2 v(x,t)}{\partial x^2} = -r\frac{\partial^2 v(x,t)}{\partial t^2} \quad 0 < x < L, \;\; t > 0 \\
&v(0,t) = v(L,t) = 0 \quad t \geq 0 \\
&v(x,0) = v_0(x), \quad \frac{\partial v}{\partial t}(x,0) = v_1(x) \quad 0 \leq x \leq L
\end{aligned}
$$

with $p > 0$ the tension of the string and r its density.

The transverse vibration of a string

We seek separated solutions of the form

$$v(x,t) = u(x)w(t)$$

hence

$$-p\frac{d^2u(x)}{dx^2}w(t) = -ru(x)\frac{d^2w(t)}{dt^2}$$

or

$$\frac{-p\frac{d^2u(x)}{dx^2}}{ru(x)} = \frac{\frac{d^2w(t)}{dt^2}}{w(t)} \quad 0 < x < L,\ \ t \geq 0$$

Both sides equal a constant λ.

The transverse vibration of a string

We seek separated solutions of the form

$$v(x,t) = u(x)w(t)$$

hence

$$-p\frac{d^2u(x)}{dx^2}\,w(t) = -ru(x)\frac{d^2w(t)}{dt^2}$$

or

$$\frac{-p\dfrac{d^2u(x)}{dx^2}}{ru(x)} = \frac{\dfrac{d^2w(t)}{dt^2}}{w(t)} \quad 0 < x < L,\;\; t \geq 0$$

Both sides equal a constant λ.
So we are led to seek a number λ and a function $u(x) \neq 0$ so that

$$-\,p\frac{d^2u(x)}{dx^2} = \lambda ru(x) \quad 0 < x < L$$

$$u(0) = u(L) = 0$$

Associated eigenvalue problem

$$
\begin{aligned}
- C^2 u''(x) &= \lambda u(x), \quad 0 < x < L \\
u(0) = u(L) &= 0
\end{aligned}
$$

where $C^2 = p/r$.
Then the solution of the problem is:

$$
\begin{cases}
\lambda^{(k)} = \dfrac{k^2 C^2 \pi^2}{L^2} \\
u^{(k)}(x) = \sqrt{2/L}\sin\left(\dfrac{k\pi x}{l}\right)
\end{cases}
$$

Vibrating membrane

Consider the small, transverse vibration of a thin membrane stretched over a bounded region $\Omega \subset \mathbb{R}^2$ and fixed at the boundary $\partial\Omega$
The vertical displacement $v(x, y, t)$ satisfies

$$
\begin{aligned}
-\Delta v &= -\frac{\partial^2 v}{\partial t^2} \quad (x, y) \in \Omega,\ \ t > 0 \\
v(x, y, t) &= 0 \quad (x, y) \in \partial\Omega, \quad t \geq 0 \\
&\text{initial condition}
\end{aligned}
$$

Using again separation of variables $v(x, y, t) = u(x, y)w(t)$ we are led to the eigenvalue problem: find λ and $u(x, y) \neq 0$ such that

$$-\Delta u = \lambda u \ \ (x, y) \in \Omega, \qquad u(x, y) = 0 \ \ (x, y) \in \partial\Omega$$

Other similar examples

- Heat conduction
- Vibration of an elastic solid

Maxwell's equations

The classical electromagnetic field is described by the four vectors $\mathcal{E}$, $\mathcal{D}$, $\mathcal{H}$, and $\mathcal{B}$ which are functions of the position $\mathbf{x} \in \mathbb{R}^3$ and of the time $t \in \mathbb{R}$. The vectors $\mathcal{E}$ and $\mathcal{H}$ are referred to as the electric and magnetic field, while $\mathcal{D}$ and $\mathcal{B}$ are the electric and magnetic displacements, respectively.
The Faraday law of induction states

$$\int_{\partial\Sigma} \mathcal{E} \cdot ds = -\frac{d}{dt} \int_{\Sigma} \mathcal{B} \cdot \mathbf{n}$$

The Ampère law says

$$\int_{\partial\Sigma} \mathcal{H} \cdot ds = \frac{d}{dt} \int_{\Sigma} \mathcal{D} \cdot \mathbf{n} + \int_{\Sigma} \mathcal{J} \cdot \mathbf{n}$$

where $\mathcal{J}$ denotes the current density vector.

It can be noticed that the fields $\mathcal{E}$, $\mathcal{H}$, $\mathcal{B}$, and $\mathcal{D}$ have a different nature. Indeed, the first two are integral 1-forms, while the latter two are integral 2-forms.
This remark is of fundamental importance for the design of finite element schemes.

It can be noticed that the fields $\mathcal{E}$, $\mathcal{H}$, $\mathcal{B}$, and $\mathcal{D}$ have a different nature. Indeed, the first two are integral 1-forms, while the latter two are integral 2-forms.
This remark is of fundamental importance for the design of finite element schemes.
The differential forms of Faraday and Ampère laws read

$$\frac{\partial \mathcal{B}}{\partial t} + \mathbf{curl}\,\mathcal{E} = 0$$
$$\frac{\partial \mathcal{D}}{\partial t} - \mathbf{curl}\,\mathcal{H} = -\mathcal{J}$$

It can be noticed that the fields $\mathcal{E}$, $\mathcal{H}$, $\mathcal{B}$, and $\mathcal{D}$ have a different nature. Indeed, the first two are integral 1-forms, while the latter two are integral 2-forms.
This remark is of fundamental importance for the design of finite element schemes.
The differential forms of Faraday and Ampère laws read

$$\frac{\partial \mathcal{B}}{\partial t} + \mathbf{curl}\,\mathcal{E} = 0$$
$$\frac{\partial \mathcal{D}}{\partial t} - \mathbf{curl}\,\mathcal{H} = -\mathcal{J}$$

which are usually referred to as Maxwell's equations, together with the two Gauss Laws

$$\operatorname{div}\mathcal{D} = \rho$$
$$\operatorname{div}\mathcal{B} = 0$$

where ρ denotes the charge density function.

The time-harmonic Maxwell system is considered, for instance, when the Fourier transform in time is used or when the propagation of electromagnetic waves at a given frequency is studied. Then, given a frequency ω, we consider the *ansatz*:

$$\mathcal{E}(\mathbf{x},t)=\Re\left(e^{-i\omega t}\mathbf{E}(\mathbf{x})\right)$$
$$\mathcal{D}(\mathbf{x},t)=\Re\left(e^{-i\omega t}\mathbf{D}(\mathbf{x})\right)$$
$$\mathcal{H}(\mathbf{x},t)=\Re\left(e^{-i\omega t}\mathbf{H}(\mathbf{x})\right)$$
$$\mathcal{B}(\mathbf{x},t)=\Re\left(e^{-i\omega t}\mathbf{B}(\mathbf{x})\right)$$

where $\Re$ denotes the *real part*. We define also

$$\mathcal{J}(\mathbf{x},t)=\Re\left(e^{-i\omega t}\mathbf{J}(\mathbf{x})\right)$$
$$\rho(\mathbf{x},t)=\Re\left(e^{-i\omega t}r(\mathbf{x})\right)$$

Standard constitutive equations for linear media read

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

where ε and μ denote the electric permittivity and the magnetic permeability, respectively. For general inhomogeneous, anisotropic materials ε and μ are 3×3 positive definite matrix functions.

Standard constitutive equations for linear media read

$$\mathbf{D} = \varepsilon \mathbf{E} \qquad \mathbf{B} = \mu \mathbf{H}$$

where ε and μ denote the electric permittivity and the magnetic permeability, respectively. For general inhomogeneous, anisotropic materials ε and μ are 3×3 positive definite matrix functions.

Inserting constitutive relations into the Maxwell's equations and considering the time-harmonic assumptions, we get the time harmonic Maxwell's equations

$$\begin{aligned}
&\mathbf{curl}\, \mathbf{E} - i\omega\mu \mathbf{H} = \mathbf{0} \\
&\operatorname{div}(\varepsilon \mathbf{E}) = r \\
&\mathbf{curl}\, \mathbf{H} + i\omega\varepsilon \mathbf{E} = \mathbf{J} \\
&\operatorname{div}(\mu \mathbf{H}) = 0
\end{aligned}$$

It is a standard procedure to eliminate one variable and to write the time harmonic Maxwell's system as a second order system. Eliminating for instance the field $\mathbf{H}$, we get

$$\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\varepsilon\mathbf{E} = \mathbf{F}$$

where $\mathbf{F}$ is given by $i\omega\mathbf{J}$, together with the divergence condition (which follows from the equation)

$$-\omega^2\,\mathrm{div}(\varepsilon\mathbf{E}) = \mathrm{div}\,\mathbf{F}.$$

It is a standard procedure to eliminate one variable and to write the time harmonic Maxwell's system as a second order system. Eliminating for instance the field $\mathbf{H}$, we get

$$\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\varepsilon\mathbf{E} = \mathbf{F}$$

where $\mathbf{F}$ is given by $i\omega\mathbf{J}$, together with the divergence condition (which follows from the equation)

$$-\omega^2\,\mathrm{div}(\varepsilon\mathbf{E}) = \mathrm{div}\,\mathbf{F}.$$

The equation is usually equipped with suitable boundary conditions. The simplest one is the perfect conducting boundary condition which reads

$$\mathbf{E}\times\mathbf{n} = 0$$

where $\mathbf{n}$ is the outward unit vector.

It is a standard procedure to eliminate one variable and to write the time harmonic Maxwell's system as a second order system. Eliminating for instance the field $\mathbf{H}$, we get

$$\mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}) - \omega^2\varepsilon\mathbf{E} = \mathbf{F}$$

where $\mathbf{F}$ is given by $i\omega\mathbf{J}$, together with the divergence condition (which follows from the equation)

$$-\omega^2\,\mathrm{div}(\varepsilon\mathbf{E}) = \mathrm{div}\,\mathbf{F}.$$

The equation is usually equipped with suitable boundary conditions. The simplest one is the perfect conducting boundary condition which reads

$$\mathbf{E}\times\mathbf{n} = 0$$

where $\mathbf{n}$ is the outward unit vector.

The solvability of the above system is strictly related to the frequency value ω. In particular, it is ill-posed if $\omega^2 = \lambda$ where λ satisfies

$$\begin{cases} \mathbf{curl}(\mu^{-1}\,\mathbf{curl}\,\mathbf{E}) = \lambda\varepsilon\mathbf{E} & \text{in } \Omega \\ \mathrm{div}(\varepsilon\mathbf{E}) = 0 & \text{in } \Omega \\ \mathbf{E}\times\mathbf{n} = 0 & \text{on } \partial\Omega \end{cases}$$

Two alternative mixed formulations

We observe that $\lambda = 0$ is associated to the infinite dimensional space $\nabla H^1_0(\Omega)$.

Two alternative mixed formulations

We observe that $\lambda = 0$ is associated to the infinite dimensional space $\nabla H^1_0(\Omega)$.
To impose $\lambda \neq 0$ we introduce two different mixed formulations.
Maxwell (1) Set $\mathbf{u} = \mathbf{E}$, let $\psi =$ be the Lagrange multiplier associated to div free condition

Kikuchi 1989

Find $(\lambda, \mathbf{u}) \in \mathbb{R} \times \mathbf{H}_0(\mathbf{curl}; \Omega))$ such that for some $\psi \in H^1_0(\Omega)$

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,\psi, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl}; \Omega) \\ (\mathbf{grad}\,\phi, \mathbf{u}) = 0 & \forall \phi \in H^1_0(\Omega) \end{cases}$$

Two alternative mixed formulations

We observe that $\lambda = 0$ is associated to the infinite dimensional space $\nabla H^1_0(\Omega)$.
To impose $\lambda \neq 0$ we introduce two different mixed formulations.
Maxwell (1) Set $\mathbf{u} = \mathbf{E}$, let $\psi =$ be the Lagrange multiplier associated to div free condition

Kikuchi 1989

Find $(\lambda, \mathbf{u}) \in \mathbb{R} \times \mathbf{H}_0(\mathbf{curl};\Omega))$ such that for some $\psi \in H^1_0(\Omega)$

$$\begin{cases} (\mathbf{curl}\,\mathbf{u}, \mathbf{curl}\,\mathbf{v}) + (\mathbf{grad}\,\psi, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_0(\mathbf{curl};\Omega) \\ (\mathbf{grad}\,\phi, \mathbf{u}) = 0 & \forall \phi \in H^1_0(\Omega) \end{cases}$$

Maxwell (2) Set $\boldsymbol{\sigma} = \mathbf{E}$ and $\mathbf{z} = \lambda^{-1}\,\mathbf{curl}\,\mathbf{E}$

Boffi-Fernandes-G.-Perugia 1999
Boffi 2000, 2001

Find $(\lambda, \boldsymbol{\sigma}) \in \mathbb{R} \times \mathbf{H}_0(\mathbf{curl};\Omega))$ such that for some $\mathbf{z} \in \mathbf{H}_0(\mathrm{div}^0;\Omega)$

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\tau}) + (\mathbf{curl}\,\boldsymbol{\tau}, \mathbf{z}) = 0 & \forall \boldsymbol{\tau} \in \mathbf{H}_0(\mathbf{curl};\Omega) \\ (\mathbf{curl}\,\boldsymbol{\sigma}, \mathbf{w}) = -\lambda(\mathbf{z}, \mathbf{w}) & \forall \mathbf{w} \in \mathbf{H}_0(\mathrm{div}^0;\Omega) \end{cases}$$

Commuting diagram property (de Rham complex)

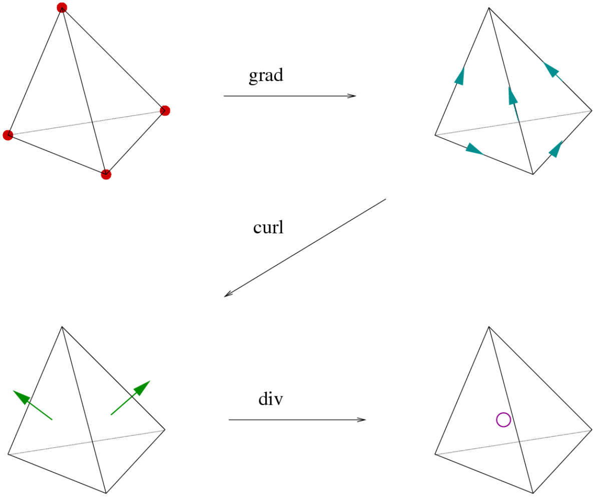
Douglas–Roberts '82
Bossavit '88
Arnold '02
Arnold–Falk–Winther '10

$Q \subset H_0^1,\ V \subset \mathbf{H}_0(\mathbf{curl}),\ U \subset \mathbf{H}_0(\mathrm{div}),\ S \subset L^2/\mathbb{R}$

$$
\begin{array}{ccccccccc}
0 \to & Q & \xrightarrow{\nabla} & V & \xrightarrow{\mathbf{curl}} & U & \xrightarrow{\mathrm{div}} & S & \to 0 \\
& \big\downarrow \Pi_k^Q & & \big\downarrow \Pi_k^V & & \big\downarrow \Pi_k^U & & \big\downarrow \Pi_k^S & \\
0 \to & Q_k & \xrightarrow{\nabla} & V_k & \xrightarrow{\mathbf{curl}} & U_k & \xrightarrow{\mathrm{div}} & S_k & \to 0
\end{array}
$$

- Maxwell (1) or Kikuchi formulation uses Q and V
- Maxwell (2) formulation uses V and U
- U and S are used for Darcy flow or mixed Laplacian

Lowest order finite elements



Outline

Laplace eigenvalue problem

1D Laplacian

$$\begin{cases} -u''(x) = \lambda u(x) & x \in [0, \pi] \\ u(0) = u(\pi) = 0 \end{cases}$$

Exact solution:

$\lambda^{(k)} = k^2$

$u^{(k)}(t) = \sin(kt)$ $\qquad (k = 1, 2, 3, \dots)$

Laplace eigenvalue problem

1D Laplacian

$$\begin{cases} -u''(x) = \lambda u(x) & x \in [0,\pi] \\ u(0) = u(\pi) = 0 \end{cases}$$

Exact solution:

$\lambda^{(k)} = k^2$

$u^{(k)}(t) = \sin(kt)$ $\quad (k = 1, 2, 3, \dots)$

Weak form of the eigenvalue problem

Given $\Omega =]0,\pi[$, find **eigenvalues** $\lambda \in \mathbb{R}$ and non-vanishing **eigenfunctions** $u \in H_0^1(0,\pi)$ such that

$$\int_0^\pi u'(x)v'(x)\,dx = \lambda \int_0^\pi u(x)v(x)\,dx \quad \forall v \in H_0^1(0,\pi)$$

Finite element discretization

Let $V_h \subset H_0^1(0,\pi)$ finite dimensional, $V_h = \mathrm{span}\{\varphi_1, \ldots, \varphi_N\}$

Discrete problem

Find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$ with $u_h \neq 0$ such that

$$\int_0^\pi u_h'(x) v'(x)\, dx = \lambda_h \int_0^\pi u_h(x) v(x)\, dx \quad \forall v \in V_h$$

This gives the algebraic problem of the form

$$\mathsf{A}\mathsf{x} = \lambda \mathsf{M}\mathsf{x}$$

with stiffness matrix $\mathsf{A} = \{a_{ij}\}_{i,j=1}^N$ and mass matrix $\mathsf{M} = \{m_{ij}\}_{i,j=1}^N$ where

$$a_{ij} = \int_0^\pi \varphi_j'(x)\varphi_i'(x)\, dx \quad m_{ij} = \int_0^\pi \varphi_j(x)\varphi_i(x)\, dx$$

Piecewise linear finite elements

Given a uniform partition of $[0,\pi]$ of size $h = \frac{\pi}{N+1}$, let V_h be standard conforming $\mathcal{P}_1$ finite element space, then for $i,j = 1,\dots,N$

$$a_{ij} = \frac{1}{h}\begin{cases} 2 & \text{for } i = j \\ -1 & \text{for } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases} \qquad m_{ij} = \frac{h}{6}\begin{cases} 4 & \text{for } i = j \\ 1 & \text{for } |i-j| = 1 \\ 0 & \text{otherwise} \end{cases}$$

The discrete eigenmodes can be explicitly computed for any $k \in \mathbb{N}$ as

$$\lambda_h^{(k)} = \frac{6}{h^2}\frac{1-\cos(kh)}{2+\cos(kh)}$$

$$u_h^{(k)}(ih) = \sin(kih),\ \ i = 0,\dots,N+1 \qquad \text{interpolant of the continuous solution}$$

It is then immediate to deduce the optimal estimates as $h \to 0$

$$\|u^{(k)} - u_h^{(k)}\|_{H^1_0(0,\pi)} = O(h), \qquad |\lambda^{(k)} - \lambda_h^{(k)}| = O(h^2)$$

Solving the generalized eigenvalue problem with Matlab

We used Matlab for the next numerical experiments.

Since we are mainly interested to the smallest eigenvalues we can use `eigs`.
We take a diagonal matrix A with the first k entries equal to 1, and the next ones equal to $k+1,\dots,n$.

`eigs` does not always work correctly.

Solving the generalized eigenvalue problem with Matlab

We used Matlab for the next numerical experiments.

Since we are mainly interested to the smallest eigenvalues we can use `eigs`.
We take a diagonal matrix A with the first k entries equal to 1, and the next ones equal to $k+1, \ldots, n$.

`eigs` does not always work correctly.

For example, set $n = 25$, and $k = 10$. We get

exact	with `eig`	with `eigs`
1	1	1
1	1	1
1	1	1
1	1	1
1	1	1
1	1	1
1	1	1
1	1	11
1	1	12
1	1	13
11	11	14

Approximation with p/w linear finite elements

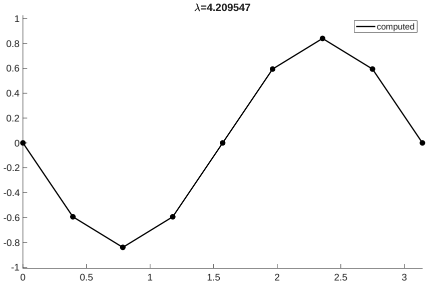
	$n = 8$	$n = 16$	$n = 32$
1	1.0129160450588	1.0032168743567	1.0008034482562
4	4.2095474481529	4.0516641802355	4.0128674974272
9	10.0802909335883	9.2631305555446	9.0652448637285
16	19.4536672593288	16.8381897926118	16.2066567209423
25	33.2628304890884	27.0649225609802	25.5059230069702

Approximation with p/w linear finite elements

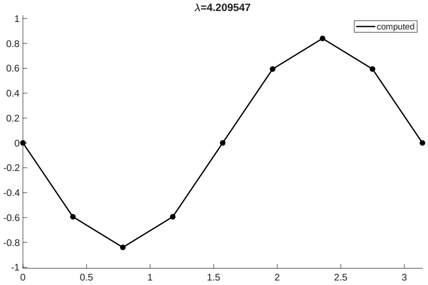
	$n = 8$	$n = 16$	$n = 32$
1	1.0129160450588	1.0032168743567	1.0008034482562
4	4.2095474481529	4.0516641802355	4.0128674974272
9	10.0802909335883	9.2631305555446	9.0652448637285
16	19.4536672593288	16.8381897926118	16.2066567209423
25	33.2628304890884	27.0649225609802	25.5059230069702

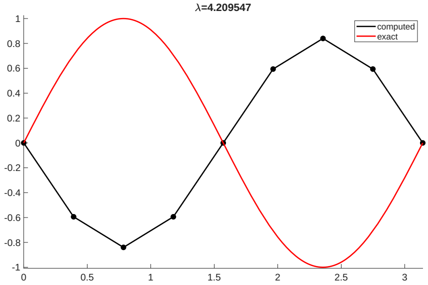
	$n = 64$	$n = 128$	$n = 256$
1	1.0002008137390	1.0000502004122	1.0000125499161
4	4.0032137930241	4.0008032549556	4.0002008016414
9	9.0162763381719	9.0040668861371	9.0010165838380
16	16.0514699897078	16.0128551720960	16.0032130198251
25	25.1257489536113	25.0313903532369	25.0078446408520

Second eigenfunction with $N = 8$

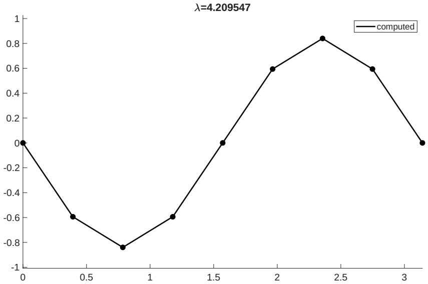


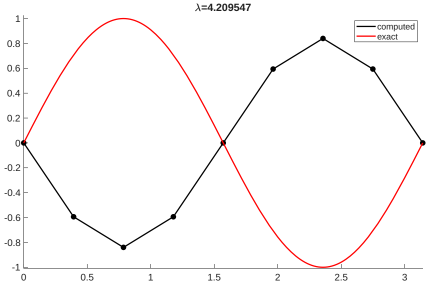
Second eigenfunction with $N = 8$

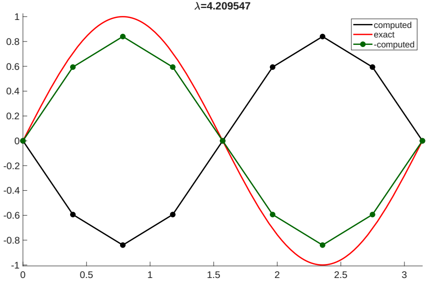




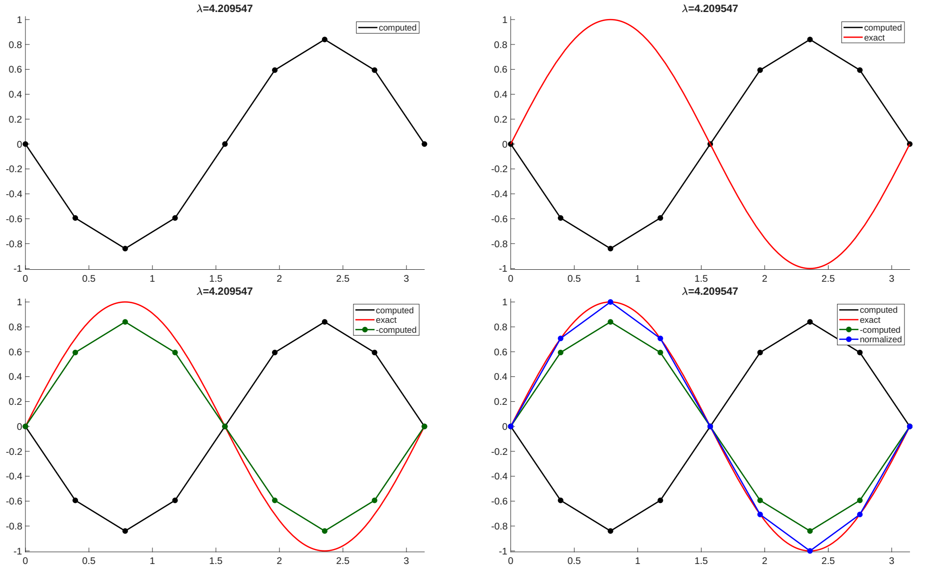
Second eigenfunction with $N = 8$







Second eigenfunction with $N = 8$



Approximation with quadratic finite elements

	Computed eigenvalue (rate)				
	$n = 8$	$n = 16$	$n = 32$	$n = 64$	$n = 128$
1	1.0000	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)	1.0000 (4.0)
4	4.0020	4.0001 (4.0)	4.0000 (4.0)	4.0000 (4.0)	4.0000 (4.0)
9	9.0225	9.0015 (3.9)	9.0001 (4.0)	9.0000 (4.0)	9.0000 (4.0)
16	16.1204	16.0082 (3.9)	16.0005 (4.0)	16.0000 (4.0)	16.0000 (4.0)
25	25.4327	25.0307 (3.8)	25.0020 (3.9)	25.0001 (4.0)	25.0000 (4.0)
36	37.1989	36.0899 (3.7)	36.0059 (3.9)	36.0004 (4.0)	36.0000 (4.0)
49	51.6607	49.2217 (3.6)	49.0148 (3.9)	49.0009 (4.0)	49.0001 (4.0)
64	64.8456	64.4814 (0.8)	64.0328 (3.9)	64.0021 (4.0)	64.0001 (4.0)
81	95.7798	81.9488 (4.0)	81.0659 (3.8)	81.0042 (4.0)	81.0003 (4.0)
100	124.9301	101.7308 (3.8)	100.1229 (3.8)	100.0080 (3.9)	100.0005 (4.0)
#	15	31	63	127	255

Remark

Convergence from above

Double order of convergence (w.r.t. approximation properties in energy norm)

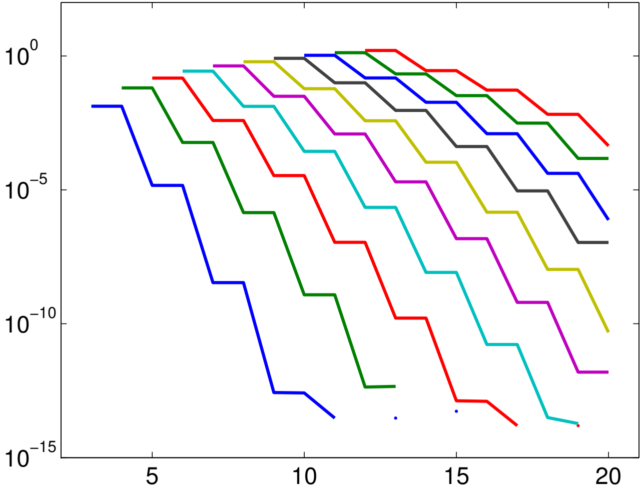
Spectral elements (p version of FEM)

Exponential convergence

Error for the eigenvalues in semilog scale

log of the error vs. number of DOFs

Each line corresponds to an eigenvalue ($\lambda = 1$ on the left)



Spectral method (p version of FEM)

Convergence of fifth eigenvalue

p	DOF	Computed
7	5	35.5593555378041
8	6	35.5593555378041
9	7	25.7779168651921
10	8	25.7779168651921
11	9	25.0306605127133
12	10	25.0306605127132
13	11	25.0004945052929
14	12	25.0004945052929
15	13	25.0000037734250
16	14	25.0000037734250
17	15	25.0000000156754
18	16	25.0000000156756
19	17	25.0000000000389
20	18	25.0000000000389

Multiple eigenvalues assocaited to eigenfunction with different regularity

This numerical example can be found in the Babuška–Osborn survey
Find eigenvalues λ and eigenfunctions $u : (-\pi, \pi) \to \mathbb{R}$ such that

$$
\begin{aligned}
&-\left(\frac{1}{\varphi'(x)}u'(x)\right)' = \lambda\varphi'(x)u(x) \quad \text{in } (-\pi,\pi)\\
&u(-\pi) = u(\pi)\\
&\frac{1}{\varphi'(-\pi)}u'(-\pi) = \frac{1}{\varphi'(\pi)}u'(\pi)
\end{aligned}
$$

where the function $\varphi(x) : (-\pi, \pi) \to \mathbb{R}$ is defined as

$$\varphi(x) = \pi^{-\alpha}|x|^{1+\alpha}\,\mathrm{sign}(x), \qquad 0 < \alpha < 1.$$

The solutions are given by $\lambda = 0$ associated to $u(x) \equiv 1$ and double eigenvalues $\lambda = k^2$ $(k = 1, 2, \dots)$ associated to eigenfunctions $u(x) = \sin(k\varphi(x))$ and $u(x) = \cos(k\varphi(x))$

The solutions are given by $\lambda = 0$ associated to $u(x) \equiv 1$ and double eigenvalues $\lambda = k^2$ $(k = 1, 2, \dots)$ associated to eigenfunctions $u(x) = \sin(k\varphi(x))$ and $u(x) = \cos(k\varphi(x))$
Let V_φ^β be the Hilbert space associated with the semi-norm

$$|v|^2_{V_\varphi^\beta} = \int_{-\pi}^{\pi} \frac{(D^\beta u)^2}{\varphi'}\, dx$$

where $D^\beta v$ is the derivative of order β

The solutions are given by $\lambda = 0$ associated to $u(x) \equiv 1$ and double eigenvalues $\lambda = k^2$ $(k = 1, 2, \dots)$ associated to eigenfunctions $u(x) = \sin(k\varphi(x))$ and $u(x) = \cos(k\varphi(x))$
Let V_φ^β be the Hilbert space associated with the semi-norm

$$|v|^2_{V_\varphi^\beta} = \int_{-\pi}^{\pi} \frac{(D^\beta u)^2}{\varphi'}\, dx$$

where $D^\beta v$ is the derivative of order β
Then

$$u_s(x) = \sin(k\varphi(x)) \in V_\varphi^\beta \qquad \forall \beta < \frac{3+\alpha}{2}$$
$$u_c(x) = \cos(k\varphi(x)) \in V_\varphi^\beta \qquad \forall \beta < \frac{5+3\alpha}{2}$$

Chosen $V_h \subset V$, the following approximation properties are satisfied:

$$\inf_{v \in V_h} \|u - v\|_V \leq Ch^s \|u\|_{V_\varphi^\beta}$$

with $s = \min\{\beta - 1, k\}$, being k the maximum degree of the polynomials contained in V_h. In particular, with linear elements we have

$$\inf_{v \in V_h} \|u_s - v\|_V \simeq h^{\frac{1+\alpha}{2} - \varepsilon}$$
$$\inf_{v \in V_h} \|u_c - v\|_V \simeq h$$

and for higher order approximations

$$\inf_{v \in V_h} \|u_s - v\|_V \simeq h^{\frac{1+\alpha}{2} - \varepsilon}$$
$$\inf_{v \in V_h} \|u_c - v\|_V \simeq h^{\frac{3(1+\alpha)}{2} - \varepsilon}.$$

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	1.7e-03	4.2e-04 (2.00)	1.0e-04 (2.00)	2.6e-05 (2.00)	6.6e-06 (1.98)
1	4.6e-03	1.3e-03 (1.88)	3.4e-04 (1.88)	9.2e-05 (1.88)	2.5e-05 (1.88)
4	6.2e-03	1.5e-03 (2.00)	3.9e-04 (2.00)	9.7e-05 (2.00)	2.4e-05 (2.00)
4	9.0e-03	2.4e-03 (1.94)	6.2e-04 (1.94)	1.6e-04 (1.93)	4.2e-05 (1.93)
9	1.4e-02	3.4e-03 (2.00)	8.5e-04 (2.00)	2.1e-04 (2.00)	5.3e-05 (2.00)
9	1.7e-02	4.2e-03 (1.97)	1.1e-03 (1.96)	2.8e-04 (1.96)	7.1e-05 (1.96)
16	2.4e-02	6.0e-03 (2.00)	1.5e-03 (2.00)	3.8e-04 (2.00)	9.4e-05 (2.00)
16	2.7e-02	6.8e-03 (1.98)	1.7e-03 (1.98)	4.4e-04 (1.98)	1.1e-04 (1.97)
25	3.7e-02	9.4e-03 (2.00)	2.3e-03 (2.00)	5.9e-04 (2.00)	1.5e-04 (2.00)
25	4.0e-02	1.0e-02 (1.99)	2.6e-03 (1.99)	6.5e-04 (1.98)	1.6e-04 (1.98)
DOF	64	128	256	512	1024

Error in the eigenvalues computed with linear elements and $\alpha = 0.9$.

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	1.2e-03	3.0e-04 (2.00)	7.4e-05 (2.00)	1.9e-05 (2.00)	4.7e-06 (2.00)
1	4.3e-03	1.4e-03 (1.59)	4.8e-04 (1.57)	1.7e-04 (1.55)	5.7e-05 (1.53)
4	4.5e-03	1.1e-03 (2.00)	2.8e-04 (2.00)	7.0e-05 (2.00)	1.8e-05 (2.00)
4	7.5e-03	2.2e-03 (1.75)	6.8e-04 (1.71)	2.2e-04 (1.67)	7.0e-05 (1.63)
9	1.0e-02	2.5e-03 (2.00)	6.2e-04 (2.00)	1.6e-04 (2.00)	3.9e-05 (2.00)
9	1.3e-02	3.6e-03 (1.86)	1.0e-03 (1.81)	3.0e-04 (1.77)	9.1e-05 (1.72)
16	1.8e-02	4.4e-03 (2.00)	1.1e-03 (2.00)	2.7e-04 (2.00)	6.9e-05 (2.00)
16	2.1e-02	5.5e-03 (1.91)	1.5e-03 (1.88)	4.2e-04 (1.84)	1.2e-04 (1.80)
25	2.7e-02	6.8e-03 (2.00)	1.7e-03 (2.00)	4.3e-04 (2.00)	1.1e-04 (2.00)
25	3.0e-02	7.9e-03 (1.94)	2.1e-03 (1.91)	5.7e-04 (1.88)	1.6e-04 (1.85)
DOF	64	128	256	512	1024

Error in the eigenvalues computed with linear elements and $\alpha = 0.5$.

Exact	Relative error (rate)				
	$N = 64$	$N = 128$	$N = 256$	$N = 512$	$N = 1024$
1	8.5e-04	2.1e-04 (2.00)	5.3e-05 (2.00)	1.3e-05 (2.00)	3.3e-06 (2.00)
1	1.3e-03	4.2e-04 (1.59)	1.6e-04 (1.45)	6.2e-05 (1.33)	2.6e-05 (1.24)
4	3.4e-03	8.4e-04 (2.00)	2.1e-04 (2.00)	5.2e-05 (2.00)	1.3e-05 (2.00)
4	3.7e-03	1.0e-03 (1.85)	3.1e-04 (1.75)	1.0e-04 (1.62)	3.6e-05 (1.49)
9	7.5e-03	1.9e-03 (2.00)	4.7e-04 (2.00)	1.2e-04 (2.00)	2.9e-05 (2.00)
9	7.8e-03	2.1e-03 (1.93)	5.6e-04 (1.87)	1.6e-04 (1.78)	5.2e-05 (1.67)
16	1.3e-02	3.3e-03 (2.01)	8.3e-04 (2.00)	2.1e-04 (2.00)	5.2e-05 (2.00)
16	1.4e-02	3.5e-03 (1.96)	9.2e-04 (1.92)	2.5e-04 (1.86)	7.4e-05 (1.78)
25	2.1e-02	5.2e-03 (2.01)	1.3e-03 (2.00)	3.2e-04 (2.00)	8.1e-05 (2.00)
25	2.1e-02	5.3e-03 (1.98)	1.4e-03 (1.95)	3.7e-04 (1.91)	1.0e-04 (1.84)
DOF	64	128	256	512	1024

Error in the eigenvalues computed with linear elements and $\alpha = 0.1$.

Exact	Relative error (rate)				
	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
1	9.0e-06	5.6e-07 (4.00)	3.5e-08 (4.00)	2.2e-09 (3.97)	1.6e-10 (3.77)
1	5.0e-04	1.7e-04 (1.52)	6.1e-05 (1.50)	2.2e-05 (1.50)	7.7e-06 (1.50)
4	8.9e-05	5.6e-06 (3.99)	3.5e-07 (4.00)	2.2e-08 (4.00)	1.4e-09 (4.00)
4	5.8e-04	1.8e-04 (1.69)	6.2e-05 (1.54)	2.2e-05 (1.51)	7.7e-06 (1.50)
9	3.9e-04	2.5e-05 (3.97)	1.6e-06 (3.99)	9.9e-08 (4.00)	6.1e-09 (4.00)
9	8.7e-04	2.0e-04 (2.15)	6.3e-05 (1.66)	2.2e-05 (1.53)	7.7e-06 (1.51)
16	1.2e-03	7.5e-05 (3.95)	4.7e-06 (3.99)	3.0e-07 (4.00)	1.9e-08 (4.00)
16	1.6e-03	2.5e-04 (2.72)	6.6e-05 (1.91)	2.2e-05 (1.59)	7.7e-06 (1.52)
25	2.7e-03	1.8e-04 (3.92)	1.1e-05 (3.98)	7.1e-07 (3.99)	4.4e-08 (4.00)
25	3.1e-03	3.5e-04 (3.17)	7.2e-05 (2.27)	2.2e-05 (1.70)	7.7e-06 (1.54)
DOF	64	128	256	512	1024

Error in the eigenvalues computed with quadratic elements and $\alpha = 0.5$.

Exact	Relative error (rate)				
	$N = 32$	$N = 64$	$N = 128$	$N = 256$	$N = 512$
1	3.4e-06	2.6e-07 (3.70)	2.1e-08 (3.62)	1.8e-09 (3.54)	1.6e-10 (3.51)
1	2.0e-04	9.1e-05 (1.16)	4.2e-05 (1.11)	2.0e-05 (1.10)	9.1e-06 (1.10)
4	4.0e-05	2.7e-06 (3.88)	1.9e-07 (3.84)	1.4e-08 (3.78)	1.1e-09 (3.70)
4	2.6e-04	9.5e-05 (1.45)	4.2e-05 (1.17)	2.0e-05 (1.11)	9.1e-06 (1.10)
9	1.9e-04	1.2e-05 (3.92)	8.1e-07 (3.91)	5.5e-08 (3.88)	3.9e-09 (3.82)
9	4.4e-04	1.1e-04 (2.02)	4.3e-05 (1.32)	2.0e-05 (1.14)	9.1e-06 (1.11)
16	5.6e-04	3.7e-05 (3.93)	2.4e-06 (3.94)	1.6e-07 (3.92)	1.1e-08 (3.89)
16	8.7e-04	1.4e-04 (2.65)	4.5e-05 (1.60)	2.0e-05 (1.20)	9.1e-06 (1.12)
25	1.3e-03	8.8e-05 (3.92)	5.7e-06 (3.95)	3.7e-07 (3.95)	2.4e-08 (3.92)
25	1.7e-03	2.0e-04 (3.12)	4.9e-05 (1.99)	2.0e-05 (1.30)	9.1e-06 (1.13)
DOF	64	128	256	512	1024

Error in the eigenvalues computed with quadratic elements and $\alpha = 0.1$.

Outline

Two dimensional Laplace eigenvalue problem

Let $\Omega =]0,\pi[\times]0,\pi[$, find $\lambda \in \mathbb{R}$ and $u \neq 0$ such that

$$
\begin{aligned}
-\Delta u &= \lambda u && \text{in } \Omega \\
u &= 0 && \text{on } \partial\Omega
\end{aligned}
$$

Weak form

Find $\lambda \in \mathbb{R}$ and non-vanishing $u \in H^1_0(\Omega)$ such that

$$(\nabla u, \nabla v) = \lambda(u, v) \quad \forall v \in H^1_0(\Omega)$$

Exact solution:

$\lambda_{m,n} = m^2 + n^2$

$u_{m,n}(x, y) = \sin(mx)\sin(ny)$ $\qquad (m, n = 1, 2, 3, \dots)$

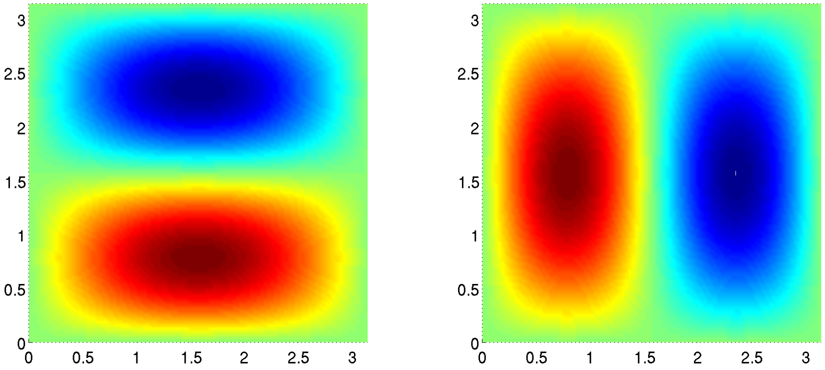
Approximation with p/w linear finite elements

Unstructured mesh

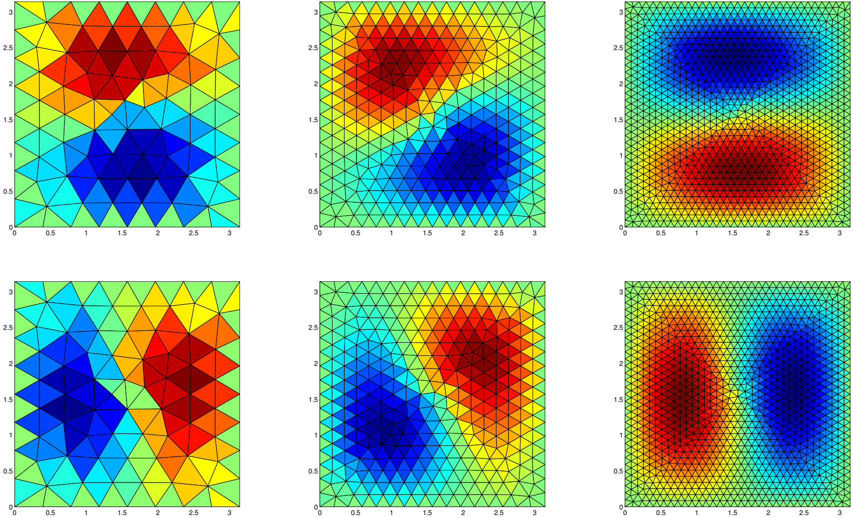
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.2468	2.0463 (2.4)	2.0106 (2.1)	2.0025 (2.1)	2.0006 (2.0)
5	6.5866	5.2732 (2.5)	5.0638 (2.1)	5.0154 (2.0)	5.0038 (2.0)
5	6.6230	5.2859 (2.5)	5.0643 (2.2)	5.0156 (2.0)	5.0038 (2.0)
8	10.2738	8.7064 (1.7)	8.1686 (2.1)	8.0402 (2.1)	8.0099 (2.0)
10	12.7165	11.0903 (1.3)	10.2550 (2.1)	10.0610 (2.1)	10.0152 (2.0)
10	14.3630	11.1308 (1.9)	10.2595 (2.1)	10.0622 (2.1)	10.0153 (2.0)
13	19.7789	14.8941 (1.8)	13.4370 (2.1)	13.1046 (2.1)	13.0258 (2.0)
13	24.2262	14.9689 (2.5)	13.4435 (2.2)	13.1053 (2.1)	13.0258 (2.0)
17	34.0569	20.1284 (2.4)	17.7468 (2.1)	17.1771 (2.1)	17.0440 (2.0)
17		20.2113	17.7528 (2.1)	17.1798 (2.1)	17.0443 (2.0)
#	9	56	257	1106	4573

Multiple eigenfunctions

Exact solutions $(5 = 1^2 + 2^2 = 2^2 + 1^2)$



Multiple eigenfunctions (discrete)

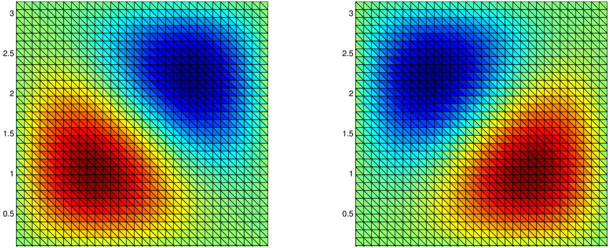


Uniform mesh

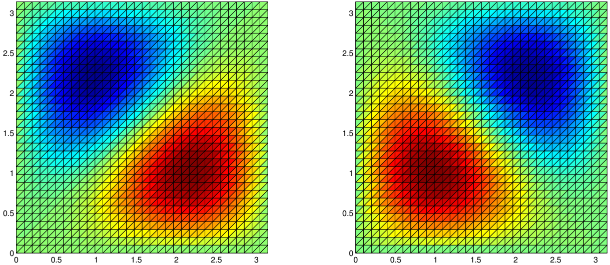
	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.3168	2.0776 (2.0)	2.0193 (2.0)	2.0048 (2.0)	2.0012 (2.0)
5	6.3387	5.3325 (2.0)	5.0829 (2.0)	5.0207 (2.0)	5.0052 (2.0)
5	7.2502	5.5325 (2.1)	5.1302 (2.0)	5.0324 (2.0)	5.0081 (2.0)
8	12.2145	9.1826 (1.8)	8.3054 (2.0)	8.0769 (2.0)	8.0193 (2.0)
10	15.5629	11.5492 (1.8)	10.3814 (2.0)	10.0949 (2.0)	10.0237 (2.0)
10	16.7643	11.6879 (2.0)	10.3900 (2.1)	10.0955 (2.0)	10.0237 (2.0)
13	20.8965	15.2270 (1.8)	13.5716 (2.0)	13.1443 (2.0)	13.0362 (2.0)
13	26.0989	17.0125 (1.7)	13.9825 (2.0)	13.2432 (2.0)	13.0606 (2.0)
17	32.4184	21.3374 (1.8)	18.0416 (2.1)	17.2562 (2.0)	17.0638 (2.0)
17		21.5751	18.0705 (2.1)	17.2626 (2.0)	17.0653 (2.0)
#	9	49	225	961	3969

Multiple eigenfunctions (uniform meshes)

Uniform mesh



Uniform mesh (reversed)



Multiple eigenfunctions (symmetric mesh)

Criss-cross mesh

Exact	Computed eigenvalue (rate)				
	$N = 4$	$N = 8$	$N = 16$	$N = 32$	$N = 64$
2	2.0880	2.0216 (2.0)	2.0054 (2.0)	2.0013 (2.0)	2.0003 (2.0)
5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
5	5.6811	5.1651 (2.0)	5.0408 (2.0)	5.0102 (2.0)	5.0025 (2.0)
8	9.4962	8.3521 (2.1)	8.0863 (2.0)	8.0215 (2.0)	8.0054 (2.0)
10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
10	12.9691	10.7578 (2.0)	10.1865 (2.0)	10.0464 (2.0)	10.0116 (2.0)
13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
13	17.1879	14.0237 (2.0)	13.2489 (2.0)	13.0617 (2.0)	13.0154 (2.0)
17	25.1471	19.3348 (1.8)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
17	38.9073	19.3348 (3.2)	17.5733 (2.0)	17.1423 (2.0)	17.0355 (2.0)
18	38.9073	19.8363 (3.5)	18.4405 (2.1)	18.1089 (2.0)	18.0271 (2.0)
20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
20	38.9073	22.7243 (2.8)	20.6603 (2.0)	20.1634 (2.0)	20.0407 (2.0)
25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
25	38.9073	28.7526 (1.9)	25.8940 (2.1)	25.2201 (2.0)	25.0548 (2.0)
DOF	25	113	481	1985	8065

Outline

Symmetric eigenvalue problems in weak form

Ingredients

- V and H real Hilbert spaces. We suppose that $V \subset H$ with dense and compact embedding.
- $a : V \times V \to \mathbb{R}$ and $b : H \times H \to \mathbb{R}$ are symmetric and continuous bilinear forms.
- We suppose that a is V-elliptic, that is, there exists $\alpha > 0$ such that

$$a(v, v) \geq \alpha \|v\|_V^2 \quad \forall v \in V$$

Eigenvalue problem in weak form

Find $\lambda \in \mathbb{R}$ and $u \in V$ with $u \neq 0$, such that

$$\text{(EIG)} \quad a(u, v) = \lambda b(u, v) \quad \forall v \in V.$$

Two dimensional Laplace eigenproblem: $V = H_0^1(\Omega)$, $H = L^2(\Omega)$, $\quad a(u, v) = (\nabla u, \nabla v)$, $b(u, v) = (u, v)$ for $u, v \in V$

Properties

Find $\lambda \in \mathbb{R}$ and $u \in V$ with $u \neq 0$, such that

$$a(u, v) = \lambda b(u, v) \quad \forall v \in V.$$

The eigenvalues are real and positive

Taking $v = u$ in the above equation, we have $a(u, u) = \lambda b(u, u)$, thus

$$\lambda = \frac{a(u, u)}{b(u, u)} \quad \text{Rayleigh quotient}$$

is the ratio of real numbers.
By ellipticity $a(u, u) \geq \alpha \|u\|_V^2$ and continuity $b(u, u) \leq C_b \|u\|_H^2 \leq C_b' \|u\|_V^2$, it holds true

$$\lambda = \frac{a(u, u)}{b(u, u)} \geq \frac{\alpha}{C_b'}$$

Solution operator $T : H \to H$

Solution operator T

Given $f \in H$, $Tf \in V$ is the unique solution of the **source problem**

$$a(Tf, v) = b(f, v) \quad \forall v \in V.$$

The operator $T : H \to H$ is **self-adjoint.**

Since the bilinear forms a and b are symmetric, for any $f, g \in H$ we have

$$b(Tf, g) = b(g, Tf) = a(Tg, Tf) = a(Tf, Tg) = b(f, Tg).$$

NB T is self-adjoint with respect to $b(\cdot, \cdot)$, not automatically with respect to the ambient $(\cdot, \cdot)_H$ unless b **is** that inner product.

Solution operator $T : H \to H$

$\boxed{T(H) \subset V \text{ implies } T \text{ is a compact operator.}}$

- The **resolvent set** $\rho(T)$ is given by the complex numbers $z \in \mathbb{C}$ such that $(zI - T)$ is bijective.
- The **spectrum** of T is $\sigma(T) = \mathbb{C} \setminus \rho(T)$.
- The spectrum $\sigma(T)$ of the compact operator T contains 0 and a countable or finite set of real numbers with a cluster point possibly only at zero.
- All positive elements of $\sigma(T)$ are eigenvalues with finite multiplicity.
- The geometric multiplicity (=algebraic multiplicity) m of μ is given by

$$m = \dim(\ker(\mu I - T))$$

- Let μ be an eigenvalue of T, that is $Tu = \mu u$, then $(\lambda = 1/\mu, u)$ is an eigensolution

$$a(u, v) = a(\frac{1}{\mu} Tu, v) = \frac{1}{\mu} a(Tu, v) = \frac{1}{\mu} b(u, v) \quad \forall v \in V$$

Eigenvalues and eigenfunctions

Let $\lambda^{(k)}$, $k \in \mathbb{N}$ be the eigenvalues of (EIG) then

$$0 < \lambda^{(1)} \leq \lambda^{(2)} \leq \cdots \leq \lambda^{(k)} \leq \cdots$$

repeated according their multiplicity.

Let $u^{(k)}$ be the corresponding eigenfunctions with

$$b(u^{(k)}, u^{(k)}) = 1, \quad a(u^{(k)}, u^{(k)}) = \lambda^{(k)},$$
$$a(u^{(m)}, u^{(n)}) = b(u^{(m)}, u^{(n)}) = 0 \quad \text{if } m \neq n$$

$E^{(k)} = \mathrm{span}(u^{(k)})$ denote the associated eigenspaces

$V = \bigoplus_{i=1}^{\infty} E^{(i)}$

Rayleigh quotient

Let $E^{(k)} = \operatorname{span}(u^{(k)})$

$$\lambda^{(1)} = \min_{v\in V} \frac{a(v,v)}{b(v,v)} \qquad u^{(1)} = \arg\min_{v\in V} \frac{a(v,v)}{b(v,v)}$$

$$\lambda^{(k)} = \min_{v\in\left(\bigoplus_{i=1}^{k-1} E^{(i)}\right)^{\perp}} \frac{a(v,v)}{b(v,v)} \qquad u^{(k)} = \arg\min_{v\in\left(\bigoplus_{i=1}^{k-1} E^{(i)}\right)^{\perp}} \frac{a(v,v)}{b(v,v)}$$

Galerkin discretization

Let $V_h \subset V$ with $\dim(V_h) = N$

Discrete eigenvalue problem

Find $\lambda_h \in \mathbb{R}$ and $u_h \in V_h$ with $u_h \neq 0$, such that

$$(\mathrm{EIG}_h) \quad a(u_h, v) = \lambda_h b(u_h, v) \quad \forall v \in V_h.$$

Discrete solution operator T_h

Given $f \in H$, $T_h f \in V_h$ is the unique solution of the **discrete source problem**

$$a(T_h f, v) = b(f, v) \quad \forall v \in V_h.$$

Since V_h is finite-dimensional, T_h is compact

Discrete eigenvalues and eigenfunctions

Let $\lambda_h^{(k)}$, $k = 1, \cdots, N$ be the eigenvalues of (EIG_h) then

$$0 < \lambda_h^{(1)} \leq \lambda_h^{(2)} \leq \cdots \leq \lambda_h^{(N)}$$

repeated according their multiplicity.

Let $u_h^{(k)}$ be the corresponding eigenfunctions with

$$b(u_h^{(k)}, u_h^{(k)}) = 1, \quad a(u_h^{(k)}, u_h^{(k)}) = \lambda_h^{(k)},$$
$$a(u_h^{(m)}, u_h^{(n)}) = b(u_h^{(m)}, u_h^{(n)}) = 0 \quad \text{if } m \neq n$$

$E_h^{(k)} = \mathrm{span}(u_h^{(k)})$ denote the associated discrete eigenspace
$V_h = \bigoplus_{i=1}^{N} E_h^{(i)}$

Rayleigh quotient (discrete)

$$\lambda_h^{(1)} = \min_{v \in V_h} \frac{a(v,v)}{b(v,v)} \qquad u_h^{(1)} = \arg\min_{v \in V_h} \frac{a(v,v)}{b(v,v)}$$

$$\lambda_h^{(k)} = \min_{v \in \left(\bigoplus_{i=1}^{k-1} E_h^{(i)} \right)^{\perp}} \frac{a(v,v)}{b(v,v)} \qquad u_h^{(k)} = \arg\min_{v \in \left(\bigoplus_{i=1}^{k-1} E_h^{(i)} \right)^{\perp}} \frac{a(v,v)}{b(v,v)}$$

Rayleigh quotient (discrete)

$$\lambda_h^{(1)} = \min_{v \in V_h} \frac{a(v,v)}{b(v,v)} \qquad u_h^{(1)} = \arg\min_{v \in V_h} \frac{a(v,v)}{b(v,v)}$$

$$\lambda_h^{(k)} = \min_{v \in \left(\bigoplus_{i=1}^{k-1} E_h^{(i)} \right)^{\perp}} \frac{a(v,v)}{b(v,v)} \qquad u_h^{(k)} = \arg\min_{v \in \left(\bigoplus_{i=1}^{k-1} E_h^{(i)} \right)^{\perp}} \frac{a(v,v)}{b(v,v)}$$

$$\boxed{\Rightarrow \lambda^{(1)} \leq \lambda_h^{(1)}}$$

min-max characterization of the eigenvalues

Courant-Fisher-Weyl

Proposition

$$\lambda^{(k)} = \min_{E \in V^{(k)}} \max_{v \in E} \frac{a(v,v)}{b(v,v)}$$

where $V^{(k)}$ denotes the set of all subspaces of V with dimension equal to k.

$\boxed{\lambda^{(k)} \ge \text{min-max}}$ Take $E = \oplus_{i=1}^{k} E^{(i)}$, $v = \sum_{i=1}^{k} \alpha_i u^{(i)}$.
From orthogonalities and normalization of eigenfunctions $a(v,v)/b(v,v) \le \lambda^{(k)}$.

$\boxed{\lambda^{(k)} \le \text{min-max}}$ Take $E = \oplus_{i=1}^{k} E^{(i)}$, then $v = u^{(k)}$ gives the maximum.
If $E \ne \oplus_{i=1}^{k} E^{(i)}$, there exists $v \in E$ orthogonal to $u^{(i)}$ for $i \le k$. Thus $a(v,v)/b(v,v) \ge \lambda^{(k)}$ which shows that $E = \oplus_{i=1}^{k} E^{(i)}$ is the optimal choice for E.

Min-max for the discrete eigenvalues

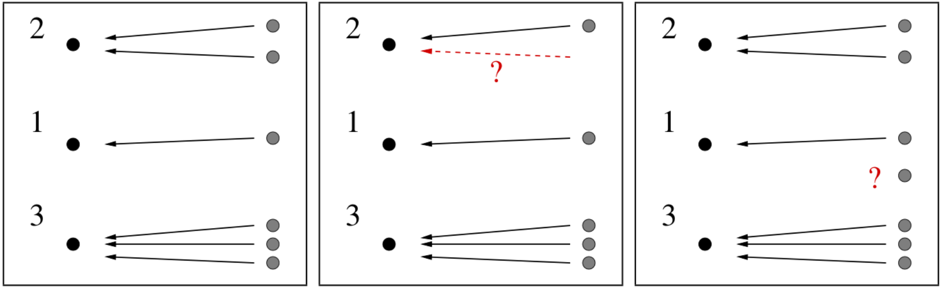
$$\lambda_h^{(k)} = \min_{E_h \in V_h^{(k)}} \max_{v \in E_h} \frac{a(v,v)}{b(v,v)}$$

where $V_h^{(k)}$ denotes the set of all subspaces of V_h with dimension equal to k.

$$\boxed{\Rightarrow \lambda^{(k)} \leq \lambda_h^{(k)} \quad \forall k \leq N}$$

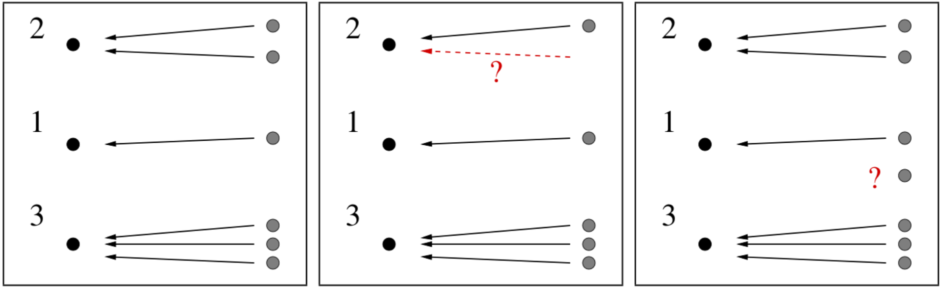
Outline

Convergence of eigenvalues



Good Bad (missing) Bad (spurious)

Convergence of eigenvalues



Good Bad (missing) Bad (spurious)

- Given an eigenvalue $\lambda^{(k)}$ of multiplicity m, there are exactly m discrete eigenvalues converging to $\lambda^{(k)}$ as $h \to 0$
- There are no spurious eigenvalues

Convergence of eigenfunctions

We cannot simply require $u_h^{(k)}$ to converge to $u^{(k)}$ in a suitable norm.

- The eigenspace associated with multiple eigenvalues can be approximated by the eigenspaces of distinct discrete eigenvalues.
- Even in the case of simple eigenvalues, the normalization of the eigenfunctions is not enough to ensure convergence, since they might have the wrong sign.