ON THE "$-$grad div $+\, s$ curl rot" OPERATOR

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Abstract

In this paper we consider the operator $-\operatorname{grad}\operatorname{div} + s\operatorname{curl}\operatorname{rot}$ in $\mathbb{R}^n$ and its finite element approximation. This operator (or a variant of it) is often present in problems which involve the penalization of the rotational free constraint. Recent results show that no standard finite element can be used to approximate this problem on general domains for some intriguing mathematical reason: the space $H^1(\Omega)^n \cap H_0(\operatorname{div};\Omega)$ is not dense in $H_0(\operatorname{div};\Omega) \cap H(\operatorname{rot};\Omega)$. In the present work we propose a new variational formulation for this operator which allows us to overcome that inconvenience.

1. Introduction

Let us consider the following problem: given $\underline{f}$ and $s > 0$, find $\underline{u}$ such that

$$
\begin{cases}
-\operatorname{grad}\operatorname{div}\underline{u} + s\operatorname{curl}\operatorname{rot}\underline{u} = \underline{f}, & \text{in } \Omega,\\
\underline{u}\cdot\underline{n} = 0 & \text{on } \partial\Omega,\\
\operatorname{rot}\underline{u} = 0 & \text{on } \partial\Omega.
\end{cases}
\tag{1}
$$

Problem (1) arises in several problems of interest. We refer the reader, for instance, to [2, 13]. Usually the term $s\operatorname{curl}\operatorname{rot}\underline{u}$ is understood as a penalization parameter.

For the sake of completeness, we say that it is possible to avoid the penalization procedure following the ideas, for instance, of [12, 11, 3].

Here, for simplicity, Ω is a polygon in $\mathbb{R}^2$. Our results apply to the three-dimensional formulation as well. Our notation is:

$$\begin{aligned} &\varphi : \Omega \to \mathbb{R}, && \underline{u} = (u_1, u_2) : \Omega \to \mathbb{R}^2, \\ &\operatorname{grad}\varphi = (\varphi_x, \varphi_y), && \operatorname{div}\underline{u} = u_{1,x} + u_{2,y}, \\ &\operatorname{curl}\varphi = (\varphi_y, -\varphi_x), && \operatorname{rot}\underline{u} = u_{2,x} - u_{1,y}. \end{aligned}$$

A variational formulation of (1) reads: find $\underline{u} \in V = H_0(\mathrm{div};\Omega) \cap H(\mathrm{rot};\Omega)$ such that

$$(\operatorname{div}\underline{u}, \operatorname{div}\underline{u}^*) + s(\operatorname{rot}\underline{u}, \operatorname{rot}\underline{u}^*) = (\underline{f}, \underline{u}^*) \quad \forall \underline{u}^* \in V, \tag{2}$$

where

$$\begin{aligned} &H_0(\mathrm{div};\Omega) = \{\underline{u} \in L^2(\Omega)^2 : \operatorname{div}\underline{u} = 0 \text{ in } \Omega,\ \underline{u}\cdot\underline{n} = 0 \text{ on } \partial\Omega\}, \\ &H(\mathrm{rot};\Omega) = \{\underline{u} \in L^2(\Omega)^2 : \operatorname{rot}\underline{u} = 0\}. \end{aligned}$$

The numerical approximation of problem (1) is a major source of troubles. Recent papers ([9, 8, 5]) show that basically no standard (conforming) finite element scheme can solve the problem. This is due to the fact that, in the case of nonconvex domains, the singular functions belonging to V but not to $H^1(\Omega)^2$ cannot be approximated in the energy norm by means of piecewise polynomials that preserve the continuity of both the normal and the tangential component.

A possible way to circumvent the problem, which has been proposed and analyzed in [1, 6], is to split the solution of (1) into its singular and regular part. The regular part can then be approximated using standard finite elements, while the singular one can be deduced by the shape of Ω. Unfortunately this approach does not extend easily to the three-dimensional case.

On the other hand, the scheme proposed in [2] (see also [10]) can be used for the approximation of the associated eigenvalue problem. However, this method does not apply to the source problem (1) because the arising matrix is singular.

For this reason, it would be advisable to find a new formulation of (2) which avoid using the space $H_0(\mathrm{div};\Omega) \cap H(\mathrm{rot};\Omega)$. This will be done in the next section. Section 3 presents a possible way of discretizing it.

In the sequel we do not give all the details of the proofs, which will be published in a forthcoming paper together with some numerical tests.

2. New variational formulation

In the previous section we observed that the main source of troubles for the approximation of problem (1) is given by the difficulties of finding a discrete space which is contained in $H_0(\mathrm{div};\Omega)\cap H(\mathrm{rot};\Omega)$ without being in $H^1(\Omega)^2$. For this reason we propose a new formulation in which the divergence and rotational operators are applied to different vector fields.

Given $\underline{f} \in L^2(\Omega)^2$ we consider the minimization problem of finding $\underline{u} \in H_0(\mathrm{div};\Omega)$, $\underline{w} \in H(\mathrm{rot};\Omega)$, $\underline{\alpha} \in L^2(\Omega)^2$ and $\underline{\beta} \in L^2(\Omega)^2$ such that

$$(\underline{u},\underline{w},\underline{\alpha},\underline{\beta}) = \arg\min\left\{\frac{1}{2}\int_\Omega |\underline{u}-\underline{w}|^2\,d\underline{x} + \frac{1}{2}\int_\Omega |\underline{\alpha}+s\underline{\beta}-\underline{f}|^2\,d\underline{x}\right\} \tag{3}$$

under the constraints

$$\underline{\alpha} = -\,\mathrm{grad}\,\mathrm{div}\,\underline{u}, \quad \underline{\beta} = \mathrm{curl}\,\mathrm{rot}\,\underline{w}. \tag{4}$$

Problem (3) is readily equivalent to problem (1). We shall impose the constraints (4) by making use of two Lagrange multipliers. Our formulation then reads: find the saddle point

$\underline{u}$, $\underline{w}$, $\underline{\alpha}$, $\underline{\beta}$ as above, $\underline{p} \in H_0(\mathrm{div};\Omega)$, $\underline{\lambda} \in H(\mathrm{rot};\Omega)$ of the functional

$$\begin{aligned}\mathcal{L}(\underline{u},\underline{w},\underline{\alpha},\underline{\beta},\underline{p},\underline{\lambda}) = & \Big\{ \frac{1}{2}\int_\Omega |\underline{u}-\underline{w}|^2\,d\underline{x} + \frac{1}{2}\int_\Omega |\underline{\alpha}+s\underline{\beta}-\underline{f}|^2\,d\underline{x} \\ & - \int_\Omega \underline{p}(\underline{\alpha}+\mathrm{grad}\,\mathrm{div}\,\underline{u})\,d\underline{x} - \int_\Omega \underline{\lambda}(\underline{\beta}-\mathrm{curl}\,\mathrm{rot}\,\underline{w})\,d\underline{x}\Big\}.\end{aligned} \tag{5}$$

Integrating by parts the last two terms of this functional and taking its first variation gives the following variational formulation: find $\underline{u} \in H_0(\mathrm{div};\Omega)$, $\underline{w} \in H(\mathrm{rot};\Omega)$, $\underline{\alpha} \in L^2(\Omega)^2$, $\underline{\beta} \in L^2(\Omega)^2$, $\underline{p} \in H_0(\mathrm{div};\Omega)$ and $\underline{\lambda} \in H(\mathrm{rot};\Omega)$ such that

$$\begin{cases} (\underline{u}-\underline{w},\underline{u}^*) + (\mathrm{div}\,\underline{p},\mathrm{div}\,\underline{u}^*) = 0 & \forall \underline{u}^* \in H_0(\mathrm{div};\Omega), \\ -(\underline{u}-\underline{w},\underline{w}^*) + (\mathrm{rot}\,\underline{\lambda},\mathrm{rot}\,\underline{w}^*) = 0 & \forall \underline{w}^* \in H(\mathrm{rot};\Omega), \\ (\underline{\alpha}+s\underline{\beta},\underline{\alpha}^*) - (\underline{p},\underline{\alpha}^*) = (\underline{f},\underline{\alpha}^*) & \forall \underline{\alpha}^* \in L^2(\Omega)^2, \\ s(\underline{\alpha}+s\underline{\beta},\underline{\beta}^*) - (\underline{\lambda},\underline{\beta}^*) = s(\underline{f},\underline{\beta}^*) & \forall \underline{\beta}^* \in L^2(\Omega)^2, \\ (\mathrm{div}\,\underline{u},\mathrm{div}\,\underline{p}^*) - (\underline{\alpha},\underline{p}^*) = 0 & \forall \underline{p}^* \in H_0(\mathrm{div};\Omega), \\ (\mathrm{rot}\,\underline{w},\mathrm{rot}\,\underline{\lambda}^*) - (\underline{\beta},\underline{\lambda}^*) = 0 & \forall \underline{\lambda}^* \in H(\mathrm{rot};\Omega). \end{cases} \tag{6}$$

Problem (6) fits within the framework of standard mixed methods. We shall make use of the following notation.

$$\begin{aligned} & V = H_0(\mathrm{div};\Omega) \times H(\mathrm{rot};\Omega) \times L^2(\Omega)^2 \times L^2(\Omega)^2, \\ & Q = H_0(\mathrm{div};\Omega) \times H(\mathrm{rot};\Omega), \\ & a(\cdot;\cdot) : V \times V \to \mathbb{R}, \\ & b(\cdot;\cdot) : V \times Q \to \mathbb{R}. \end{aligned} \tag{7}$$

The spaces V and Q are endowed with the following norms:

$$\begin{aligned} & \|(\underline{u},\underline{w},\underline{\alpha},\underline{\beta})\|_V^2 = \|\underline{u}\|_{\mathrm{div}}^2 + \|\underline{w}\|_{\mathrm{rot}}^2 + \|\underline{\alpha}\|_0^2 + \|\underline{\beta}\|_0^2 \\ & \|(\underline{p},\underline{\lambda})\|_Q^2 = \|\underline{p}\|_{\mathrm{div}}^2 + \|\underline{\lambda}\|_{\mathrm{rot}}^2, \end{aligned} \tag{8}$$

where as usual

$$\|\underline{u}\|_{\mathrm{div}}^2 = \|\underline{u}\|_0^2 + \|\,\mathrm{div}\,\underline{u}\|_0^2, \quad \|\underline{w}\|_{\mathrm{rot}}^2 = \|\underline{w}\|_0^2 + \|\,\mathrm{rot}\,\underline{w}\|_0^2.$$

The bilinear forms a and b are defined as follows:

(9)

$$a(\underline{u},\underline{w},\underline{\alpha},\underline{\beta};\underline{u}^*,\underline{w}^*,\underline{\alpha}^*,\underline{\beta}^*) = (\underline{u}-\underline{w},\underline{u}^*) - (\underline{u}-\underline{w},\underline{w}^*) + (\underline{\alpha}+s\underline{\beta},\underline{\alpha}^*) + s(\underline{\alpha}+s\underline{\beta},\underline{\beta}^*),$$
$$b(\underline{u},\underline{w},\underline{\alpha},\underline{\beta};\underline{p}^*,\underline{\lambda}^*) = (\operatorname{div}\underline{u},\operatorname{div}\underline{p}^*) - (\underline{\alpha},\underline{p}^*) + (\operatorname{rot}\underline{w},\operatorname{rot}\underline{\lambda}^*) - (\underline{\beta},\underline{\lambda}^*).$$

Then problem (6) reads

$$\tag{10} \begin{cases} a(\underline{u},\underline{w},\underline{\alpha},\underline{\beta};\underline{u}^*,\underline{w}^*,\underline{\alpha}^*,\underline{\beta}^*) + b(\underline{u}^*,\underline{w}^*,\underline{\alpha}^*,\underline{\beta}^*;\underline{p},\underline{\lambda}) = (\underline{f},\underline{\alpha}^*) + s(\underline{f},\underline{\beta}^*) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall(\underline{u}^*,\underline{w}^*,\underline{\alpha}^*,\underline{\beta}^*) \in V \\ b(\underline{u},\underline{w},\underline{\alpha},\underline{\beta};\underline{p}^*,\underline{\lambda}^*) = 0 \qquad\qquad\qquad\qquad \forall(\underline{p}^*,\underline{\lambda}^*) \in Q. \end{cases}$$

It is well-known that problem (10) is well posed if the *ellipticity on the kernel* and the *inf-sup* conditions are satisfied (see, e.g., [7]). Therefore let us introduce the kernel associated with the bilinear form b

$$\tag{11} \mathbb{K} = \left\{(\underline{u},\underline{w},\underline{\alpha},\underline{\beta}) \in V : b(\underline{u},\underline{w},\underline{\alpha},\underline{\beta};\underline{p}^*,\underline{\lambda}^*) = 0 \ \forall(\underline{p}^*,\underline{\lambda}^*) \in Q\right\}.$$

Let us give a sketch of the proof of the *ellipticity on the kernel*.

Proposition 1. *There exists a positive constant C_1, such that*

$$\tag{12} a(\underline{u},\underline{w},\underline{\alpha},\underline{\beta};\underline{u},\underline{w},\underline{\alpha},\underline{\beta}) \geq C_1\|(\underline{u},\underline{w},\underline{\alpha},\underline{\beta})\|_V^2 \quad \forall\underline{u},\underline{w},\underline{\alpha},\underline{\beta} \in \mathbb{K}.$$

Proof (sketch). The first step of the proof consists in noticing that if $(\underline{u},\underline{w},\underline{\alpha},\underline{\beta}) \in \mathbb{K}$ fulfills $a(\underline{u},\underline{w},\underline{\alpha},\underline{\beta};\underline{u},\underline{w},\underline{\alpha},\underline{\beta}) = 0$ then it holds necessarily $(\underline{u},\underline{w},\underline{\alpha},\underline{\beta}) = 0$. Namely, the identity

$$a(\underline{u},\underline{w},\underline{\alpha},\underline{\beta};\underline{u},\underline{w},\underline{\alpha},\underline{\beta}) = \|\underline{u}-\underline{w}\|_0^2 + \|\underline{\alpha}+s\underline{\beta}\|_0^2 = 0$$

yields $\underline{u} = \underline{w}$ and $\underline{\alpha}+s\underline{\beta} = 0$ in $L^2(\Omega)^2$; in particular $\underline{u}$ and $\underline{w}$ belong to $H_0(\mathrm{div};\Omega)\cap H(\mathrm{rot};\Omega)$. Using the fact that $(\underline{u},\underline{w},\underline{\alpha},\underline{\beta})$ is in the kernel $\mathbb{K}$ we have

$$(\operatorname{div}\underline{u},\operatorname{div}\underline{v}) + s(\operatorname{rot}\underline{u},\operatorname{rot}\underline{v}) = (\underline{\alpha}+s\underline{\beta},\underline{v}) = 0 \quad \forall\underline{v} \in H_0(\mathrm{div};\Omega)\cap H(\mathrm{rot};\Omega).$$

Hence $\operatorname{div} \underline{u} = 0$ and $\operatorname{rot} \underline{u} = 0$, and the boundary conditions give $\underline{u} = \underline{w} = 0$. Next, we have that $(\underline{\alpha}, \underline{p}^*) = 0$ for all $p^* \in H_0(\operatorname{div};\Omega)$. Since $H_0(\operatorname{div};\Omega)$ is dense in $L^2(\Omega)^2$, we get $\underline{\alpha} = 0$. Then also $\underline{\beta} = 0$.

Inequality (12) can then be proved by contradiction. □

We now check the inf-sup condition.

Proposition 2. *There exists a positive constant C_2 such that for any $(\underline{p}, \underline{\lambda})$ in Q one has*

$$\sup_{(\underline{u},\underline{w},\underline{\alpha},\underline{\beta})\in V} \frac{b(\underline{u}, \underline{w}, \underline{\alpha}, \underline{\beta}; \underline{p}, \underline{\lambda})}{||(\underline{u}, \underline{w}, \underline{\alpha}, \underline{\beta})||_V} \geq C_2||(\underline{p}, \underline{\lambda})||_Q. \tag{13}$$

Proof. Given $(\underline{p}, \underline{\lambda}) \in Q$ we define

$$\underline{u} = \underline{p}, \quad \underline{w} = \underline{\lambda}, \quad \underline{\alpha} = -\underline{p}, \quad \underline{\beta} = -\underline{\lambda}.$$

This gives

$$b(\underline{u}, \underline{w}, \underline{\alpha}, \underline{\beta}; \underline{p}, \underline{\lambda}) = ||\operatorname{div} \underline{p}||_0^2 + ||\underline{p}||_0^2 + ||\operatorname{rot} \underline{\lambda}||_0^2 + ||\underline{\lambda}||_0^2$$
$$||(\underline{u}, \underline{w}, \underline{\alpha}, \underline{\beta})||_V^2 = ||\underline{p}||_{\operatorname{div}}^2 + ||\underline{\lambda}||_{\operatorname{rot}}^2 + ||\underline{p}||_0^2 + ||\underline{\lambda}||_0^2$$

and the inf-sup condition (13) follows easily. □

3. Finite element approximation

We consider the following finite element approximation of problem (10). For the sake of simplicity we describe a low-order triangular method, which however generalizes in a natural manner to higher degrees.

Let RT_0 be the lowest-order Nédélec–Raviart–Thomas finite element space approximating $H(\operatorname{div};\Omega)$ and TR_0 the analogous approximation of $H(\operatorname{rot};\Omega)$ and denote by P_k the space of piecewise polynomials of degree not bigger than k.

We define

$$V_h = (RT_0 \cap H_0(\text{div};\Omega)) \times TR_0 \times P_0^2 \times P_0^2, \quad Q_h = (RT_0 \cap H_0(\text{div};\Omega)) \times TR_0 \tag{14}$$

and consider the following scheme: find $(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h; \underline{p}_h, \underline{\lambda}_h) \in V_h \times Q_h$ such that

$$\begin{cases} a(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h; \underline{u}^*, \underline{w}^*, \underline{\alpha}^*, \underline{\beta}^*) + b(\underline{u}^*, \underline{w}^*, \underline{\alpha}^*, \underline{\beta}^*; \underline{p}_h, \underline{\lambda}_h) = (\underline{f}, \underline{\alpha}^*) + s(\underline{f}, \underline{\beta}^*) \\ \hfill \forall(\underline{u}^*, \underline{w}^*, \underline{\alpha}^*, \underline{\beta}^*) \in V_h \\ b(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h; \underline{p}^*, \underline{\lambda}^*) = 0 \hfill \forall(\underline{p}^*, \underline{\lambda}^*) \in Q_h. \end{cases} \tag{15}$$

It follows from the standard theory of mixed approximation (see [7]) that the following stability hypotheses guarantee the well-posedness and the stability of the finite element scheme.

(ElKer) There exists $A > 0$ such that

$$a(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h; \underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h) \geq A||(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h)||_V^2 \quad \forall(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h) \in \mathbb{K}_h,$$

with

$$\mathbb{K}_h = \{(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h) \in V_h : b(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h; \underline{p}_h, \underline{\lambda}_h) = 0 \ \forall(\underline{p}_h, \underline{\lambda}_h) \in Q_h\},$$

(InfSup) there exists $B > 0$ such that

$$\sup_{(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h) \in V_h} \frac{b(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h; \underline{p}_h, \underline{\lambda}_h)}{||(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h)||_V} \geq B||(\underline{p}_h, \underline{\lambda}_h)||_Q \quad \forall(\underline{p}_h, \underline{\lambda}_h) \in Q_h.$$

Before dealing with hypotheses (ElKer) and (InfSup) we recall some basic properties of the spaces RT_0 and TR_0 (see, e.g., [4]).

Given $\underline{p}_h \in RT_0 \cap H_0(\text{div};\Omega)$ we can always consider a discrete Helmholtz decomposition by solving the following auxiliary problem: find $\underline{\sigma}_h \in RT_0 \cap H_0(\text{div};\Omega)$ and $u_h \in P_0 \cap L_0^2(\Omega)$ such that

$$\begin{cases} (\underline{\sigma}_h, \underline{\sigma}^*) + (\operatorname{div} \underline{\sigma}^*, u_h) = 0 & \forall \underline{\sigma}^* \in RT_0 \cap H_0(\text{div};\Omega) \\ (\operatorname{div} \underline{\sigma}_h, u^*) = (\operatorname{div} \underline{p}_h, u^*) & \forall u^* \in P_0 \cap L_0^2(\Omega). \end{cases} \tag{16}$$

Problem (16) is well-posed (see [7]) and from the second equation we have $\operatorname{div} \underline{\sigma}_h = \operatorname{div} \underline{p}_h$, which allows us to define $\operatorname{curl} \varphi_h = \underline{p}_h - \underline{\sigma}_h$. We observe that φ_h can be chosen in P_1. We then have

$$\underline{p}_h = \underline{\sigma}_h + \operatorname{curl} \varphi_h, \quad (\underline{\sigma}_h, \operatorname{curl} \varphi_h) = 0, \tag{17}$$

where the orthogonality of $\underline{\sigma}_h$ and $\operatorname{curl} \varphi_h$ follows from the first equation of (16) (set $\underline{\sigma}^* = \operatorname{curl} \varphi_h$).

Moreover, using the stability of problem (16) we have the following useful estimate

$$||\underline{\sigma}_h||_0 \leq C|| \operatorname{div} \underline{\sigma}_h||_0. \tag{18}$$

An analogous argument holds for a generic $\underline{\lambda}_h \in TR_0$. In this case we have the decomposition

$$\underline{\lambda}_h = \operatorname{grad} \psi_h + \underline{\tau}_h, \quad (\operatorname{grad} \psi_h, \underline{\tau}_h) = 0, \tag{19}$$

with $\psi_h \in P_1$ and the estimate

$$||\underline{\tau}_h||_0 \leq C|| \operatorname{curl} \underline{\tau}_h||_0. \tag{20}$$

The following propositions states that our choice for V_h and Q_h provides a stable approximation to (10). The proof of (ElKer) needs some technicalities and has been omitted.

Proposition 3. *The discrete spaces V_h and Q_h introduced in (14) satisfy the ellipticity in the kernel hypothesis (ElKer) with A independent of h.*

Proposition 4. *The discrete spaces V_h and Q_h introduced in (14) satisfy the inf-sup condition (InfSup) with B independent of h.*

Proof. Let $(\underline{p}_h, \underline{\lambda}_h)$ be given in Q_h. We can split $\underline{p}_h$ and $\underline{\lambda}_h$ using the discrete Helmholtz decompositions (17) and (19) as follows

$$\underline{p}_h = \underline{\sigma}_h + \operatorname{curl} \varphi_h, \quad \underline{\lambda}_h = \operatorname{grad} \psi_h + \underline{\tau}_h,$$

with $\varphi_h \in P_1$ and $\psi_h \in P_1$.

We then define

$$\begin{aligned}\underline{u}_h &= \underline{\sigma}_h,\\ \underline{w}_h &= \underline{\tau}_h,\\ \underline{\alpha}_h &= -\operatorname{curl}\varphi_h,\\ \underline{\beta}_h &= -\operatorname{grad}\psi_h.\end{aligned}$$

It is clear by the definition that we have

$$||(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h)||_V \leq C||(\underline{p}_h, \underline{\lambda}_h)||_Q.$$

Moreover, using the properties of the discrete Helmholtz decompositions, we get

$$b(\underline{u}_h, \underline{w}_h, \underline{\alpha}_h, \underline{\beta}_h; \underline{p}_h, \underline{\lambda}_h) = ||\operatorname{div}\underline{\sigma}_h||_0^2 + ||\operatorname{curl}\varphi_h||_0^2 + ||\operatorname{rot}\underline{\tau}_h||_0^2 + ||\operatorname{grad}\psi_h||_0^2.$$

The equality above concludes the proof together with the bounds (18) and (20). $\square$

References

[1] F. Assous, P. Ciarlet, Jr., and E. Sonnendrücker. Resolution of the Maxwell equations in a domain with reentrant corners. *RAIRO Modél. Math. Anal. Numér.*, 32(3):359–389, 1998.

[2] K.-J. Bathe, C. Nitikitpaiboon, and X. Wang. A mixed displacement-based finite element formulation for acoustic fluid-structure interaction. *Computers & Structures*, 56:225–237, 1995.

[3] I. Bermúdez, R. Durán, A. Muschietti, R. Rodríguez, and J. Solomin. Finite element vibration analysis of fluid–solid systems without spurious modes. *SIAM J. Numer. Anal.*, 32:1280–1295, 1995.

[4] D. Boffi. Fortin operator and discrete compactness for edge elements. *Numer. Math.*, 2000. Digital Object Identifier DOI 10.1007/s002110000182.

[5] D. Boffi, C. Chinosi, and L. Gastaldi. Approximation of the grad div operator in non-convex domains. *CMES*, 1:27–38, 2000.

[6] A.-S. Bonnet-Ben Dhia, C. Hazard, and S. Lohrengel. A singular field method for the solution of Maxwell's equations in polyhedral domains. *SIAM J. Appl. Math.*, 59(6):2028–2044 (electronic), 1999.

[7] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*. Springer-Verlag, New York, 1991.

[8] M. Costabel and M. Dauge. Maxwell and Lamé eigenvalues on polyhedra. *Math. Methods Appl. Sci.*, 22(3):243–258, 1999.

[9] M. Costabel and M. Dauge. Singularities of electromagnetic fields in polyhedral domains. *Arch. Ration. Mech. Anal.*, 151(3):221–276, 2000.

[10] L. Gastaldi. Mixed finite element methods in fluid structure systems. *Numer. Math.*, 74(2):153–176, 1996.

[11] F. Kikuchi. Mixed and penalty formulations for finite element analysis of an eigenvalue problem in electromagnetism. In *Proceedings of the first world congress on computational mechanics (Austin, Tex., 1986)*, volume 64, pages 509–521, 1987.

[12] F. Kikuchi. On a discrete compactness property for the Nédélec finite elements. *J. Fac. Sci., Univ. Tokyo, Sect. I A*, 36(3):479–490, 1989.

[13] X. Wang and K.-J. Bathe. On mixed elements for acoustic fluid-structure interactios. M^3AS, 7:329–344, 1996.