Applications of Maxwell equations

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Abstract

We analyze a finite element method for band structure calculations in dielectric photonic crystals. The method is based on a modification of Nedéléc edge elements. We prove convergence of approximated eigensolutions to those of the continuous problem under general assumptions on the mesh. The main result requires minimal regularity hypotheses on the coefficients.

1 Introduction

Photonic crystals are periodic structures composed of dielectric materials. The reason for the increase of the interest in this subject is that the spectrum of the Maxwell operator for such media is expected to have gaps. The presence of gaps means that there are prohibited frequencies of propagation of electromagnetic waves going through such crystals. This fact has many potential applications, for example, in optical communications, filters, lasers and microwaves. See [7, 11], for an introduction to photonic crystals, photonic band gap structures and some of their applications. The mathematical model can be written as a modified Maxwell's system with periodic boundary conditions. In recent papers [5, 6] a finite element method to approximate such problem based on a modification of Nedéléc edge element spaces was proposed. The convergence of the finite element scheme was proved under severe regularity restrictions and in the case of uniform mesh sequences. Here we present a proof which holds under minimal assumptions on the regularity of the eigensolutions and on the mesh sequences.

The outline of the paper is the following. The next section is devoted to the presentation of the problem together with some properties of the analytical framework. Section 3 contains the discretization of the problem and recalls the abstract setting under which the convergence of the eigensolutions can be proved. In the last section, the finite element spaces are described together those properties which yield the convergence.

2 Setting of the problem

We consider, in $\mathbb{R}^3$, the classical Maxwell's equations

$$\begin{aligned} \nabla\times \mathbf{E} - i\omega\mu\mathbf{H} &= 0 \\ \nabla\times \mathbf{H} + i\omega\varepsilon\mathbf{E} &= 0. \end{aligned} \tag{1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and the magnetic fields. We assume that the magnetic permeability is constant with $\mu = 1$. The dielectric tensor ε is real, symmetric and elliptic, in the sense that

$$\sum_{i,j=1}^{3} \varepsilon_{i,j}(\mathbf{x})\xi_i\xi_j \geq \varepsilon_0|\boldsymbol{\xi}|^2 \qquad \forall\boldsymbol{\xi}\in\mathbb{R}^3,\ \forall\mathbf{x}\in\mathbb{R}^3.$$

The medium is assumed to have unit periodicity on a cubic lattice. Thus denoting by $\mathbb{Z}$ the set of relative integer numbers, and defining the lattice $\Lambda = \mathbb{Z}^3$, we have

$$\varepsilon(\mathbf{x}+\mathbf{k}) = \varepsilon(\mathbf{x}) \qquad \forall\mathbf{x}\in\mathbb{R}^3 \text{ and } \forall\mathbf{k}\in\Lambda.$$

See Fig. 1 for possible structure of the medium.

Eliminating $\mathbf{E}$ from equation (1) we obtain

$$\begin{aligned} &\nabla\times\varepsilon^{-1}\,\nabla\times\mathbf{H} = \omega^2\mathbf{H} && \text{in } \mathbb{R}^3 \\ &\nabla\cdot\mathbf{H} = 0 && \text{in } \mathbb{R}^3. \end{aligned} \tag{2}$$

We define the periodic domain $\Omega = \mathbb{R}^3/\Lambda$ which can be identified with the unit cube $(0,1)^3$ with periodic boundary conditions. Let $K = [-\pi,\pi]^3$ be the first Brillouin zone. We consider the Bloch waves satisfying $\mathbf{H}(\mathbf{x}) =$

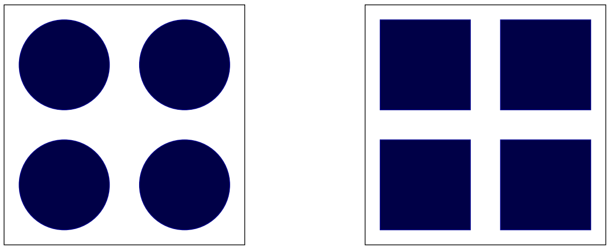

Figure 1: Four cells of circular and square rod structure. White area represents air.

$e^{i\boldsymbol{\alpha}\cdot\mathbf{x}}\mathbf{u}(\mathbf{x})$, where $\mathbf{u}$ is periodic in $\mathbf{x}$ and $\boldsymbol{\alpha} \in K$. Hence, for each $\boldsymbol{\alpha} \in K$ we look for $\mathbf{u}$, solution of the following problem

$$\begin{array}{ll} \boldsymbol{\nabla}_\alpha \times \varepsilon^{-1}\, \boldsymbol{\nabla}_\alpha \times \mathbf{u} = \omega^2 \mathbf{u} & \text{in } \Omega \\ \boldsymbol{\nabla}_\alpha \cdot \mathbf{u} = 0 & \text{in } \Omega \end{array} \tag{3}$$

Here $\boldsymbol{\nabla}_\alpha = \boldsymbol{\nabla} + i\boldsymbol{\alpha}$.

In order to introduce the variational formulation of (3) we define the periodic versions of usual Hilbert spaces. Let $\mathbf{L}^2(\Omega) = (L^2(\Omega))^3$.

$$\begin{array}{ll} H^1_{\mathrm{p}}(\Omega) & = \{v \in L^2(\Omega) :\ \boldsymbol{\nabla} v \in \mathbf{L}^2(\Omega)\} \\ \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) & = \{\mathbf{v} \in \mathbf{L}^2(\Omega) :\ \boldsymbol{\nabla}\times \mathbf{v} \in \mathbf{L}^2(\Omega)\} \\ \mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega) & = \{\mathbf{v} \in \mathbf{L}^2(\Omega) :\ \boldsymbol{\nabla}\cdot \mathbf{v} \in L^2(\Omega)\} \\ \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega) & = \{\mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega) :\ \boldsymbol{\nabla}_\alpha \cdot \mathbf{v} = 0\} \end{array} \tag{4}$$

Note that the domain Ω has no boundary. For the above definitions, functions defined on Ω are implicitly periodic and the derivative operators respect the periodicity of the domain. Enforcing the constraint in (3) by Lagrange multipliers, problem (3) can be rewritten in the following mixed form:

$$\begin{array}{l} \text{find } \omega^2 \in \mathbb{R},\ (\mathbf{u},p) \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \times H^1_{\mathrm{p}}(\Omega), \text{ with } (\mathbf{u},p) \neq (0,0), \text{ such that} \\ \left\{ \begin{array}{ll} a(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) = \omega^2(\mathbf{u},\mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \\ \overline{b(q,\mathbf{u})} = 0 & \forall q \in H^1_{\mathrm{p}}(\Omega). \end{array} \right. \end{array} \tag{5}$$

For all $\mathbf{u},\mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega)$ and $q \in H^1_{\mathrm{p}}(\Omega)$, the sesquilinear forms in (5) are defined as follows:

$$\begin{array}{l} a(\mathbf{u},\mathbf{v}) = \int_\Omega \varepsilon^{-1}\, \boldsymbol{\nabla}_\alpha \times \mathbf{u} \cdot \overline{\boldsymbol{\nabla}_\alpha \times \mathbf{v}}\, dx, \\ b(q,\mathbf{u}) = \int_\Omega \boldsymbol{\nabla}_\alpha\, q \cdot \overline{\mathbf{u}}\, dx, \\ (\mathbf{u},\mathbf{v}) = \int_\Omega \mathbf{u} \cdot \overline{\mathbf{v}}\, dx \end{array} \tag{6}$$

Moreover we introduce the kernel of $\boldsymbol{\nabla}_\alpha \cdot$, defined as follows

$$\mathbb{K} = \{\mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) : b(q,\mathbf{v}) = 0 \ \forall q \in H^1_{\mathrm{p}}(\Omega)\}. \tag{7}$$

Let $T \in \mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{L}^2(\Omega))$ be the resolvent operator associated with (5) and defined as follows. For all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $T\mathbf{f} = \mathbf{u} \in \mathbf{L}^2(\Omega)$, where $\mathbf{u}$ is the first component of the solution of the following problem:

$$\text{find } (\mathbf{u}, p) \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \times H^1_{\mathrm{p}}(\Omega), \text{ such that}$$
$$\begin{cases} a(\mathbf{u},\mathbf{v}) + b(p,\mathbf{v}) = (\mathbf{f},\mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \\ \overline{b(q,\mathbf{u})} = 0 & \forall q \in H^1_{\mathrm{p}}(\Omega). \end{cases} \tag{8}$$

If $\boldsymbol{\alpha} = (0,0,0)$, it is well-known that T is compact. In order to prove the compactness of T for all $\boldsymbol{\alpha} \in K$, let us recall some results proved in [6].

Theorem 1 *Let $\boldsymbol{\alpha} \in K$ with $\boldsymbol{\alpha} \neq (0,0,0)$. Given $\mathbf{u} \in \mathbf{L}^2(\Omega)$, there exists unique functions $\mathbf{w} \in (H^1_{\mathrm{p}}(\Omega))^3$ and $\varphi \in H^1_{\mathrm{p}}(\Omega)$ satisfying*

$$\mathbf{u} = \boldsymbol{\nabla}_\alpha \times \mathbf{w} + \boldsymbol{\nabla}_\alpha \varphi \quad \textit{with} \quad \boldsymbol{\nabla}_\alpha \cdot \mathbf{w} = 0,$$
$$\|\mathbf{w}\|_1 + \|\varphi\|_1 \leq C\|\mathbf{u}\|_0.$$

Lemma 1 *The sequence*

$$0 \to H^1_{\mathrm{p}}(\Omega) \xrightarrow{\boldsymbol{\nabla}_\alpha} \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \xrightarrow{\boldsymbol{\nabla}_\alpha \times} \mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega) \xrightarrow{\boldsymbol{\nabla}_\alpha \cdot} L^2(\Omega) \to 0 \tag{9}$$

is exact.

Lemma 2 *The operator T is compact and self-adjoint from $\mathbf{L}^2(\Omega)$ into itself.*

Proof. Thanks to Lemma 1, the second equation in (8) implies that $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$. The sesquilinear form $a(\mathbf{u},\mathbf{v})$ is hermitian, continuous and coercive on $\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$. Hence there exists a unique $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$ solution of (8). Since $\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$ is compactly embedded in $\mathbf{L}^2(\Omega)$, the operator T is compact and self-adjoint. □
As a consequence of that, T admits an increasing sequence of real, positive eigenvalues

$$0 < \omega_1^2 < \omega_2^2 < \cdots < \omega_n^2 < \cdots,$$

each associated with a finite dimensional eigenspace.

Moreover, the following regularity result holds for solutions of problem (8), see [4].

Lemma 3 *There exists $s > 1/2$, such that $\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap \mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega)$ is continuously embedded in $(H^s(\Omega))^3$.*

Moreover for all $\mathbf{f} \in \mathbf{L}^2(\Omega)$ the solution $\mathbf{u}$ of problem (8) satisfies

$$\mathbf{u} \in (H^{r+1}(\Omega))^3, \qquad \textit{with } r > 1/2. \tag{10}$$

In the above lemma the value of r depends on the ratio between the different values of ε.

3 Discretization of the problem

Let $E_h \subseteq \mathbf{H}_{\rm p}(\mathrm{curl};\Omega)$ and $Q_h \subseteq H^1_{\rm p}(\Omega)$ be finite dimensional spaces. Then the discretization of (5) reads:

$$\text{find } \omega_h^2 \in \mathbb{R},\ (\mathbf{u}_h, p_h) \in E_h \times Q_h, \text{ with } (\mathbf{u}_h, p_h) \neq (0,0), \text{ such that}$$
$$\begin{cases} a(\mathbf{u}_h,\mathbf{v}_h) + b(p_h,\mathbf{v}_h) = \omega_h^2(\mathbf{u}_h,\mathbf{v}_h) & \forall \mathbf{v}_h \in E_h \\ \overline{b(q_h,\mathbf{u}_h)} = 0 & \forall q_h \in Q_h. \end{cases} \tag{11}$$

Problem (11) can be reduced to an algebraic generalized eigenvalue problem of the form

$$\begin{pmatrix} A & B \\ B^H & 0 \end{pmatrix}\begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} = \omega_h^2 \begin{pmatrix} M & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{u} \\ \mathrm{p} \end{pmatrix} \tag{12}$$

with A the hermitian matrix associated to the sesquilinear form a, B the rectangular matrix associated to b and M the hermitian matrix associated to the scalar product in $\mathbf{L}^2(\Omega)$. To have an idea of the practical computation of the eigenvalues of this generalized eigensystem see [10].

If the matrix B has full rank, then system (12) has exactly $N(h) = \dim(E_h)$ real and positive eigenvalues:

$$0 < \omega_{1,h}^2 \leq \omega_{2,h}^2 \leq \cdots \leq \omega_{N(h),h}^2.$$

In order to analyze the convergence of the discrete eigensolutions to the continuous ones we apply the abstract theory developed in [3]. Let us first introduce the discretization of the resolvent operator $T_h : \mathbf{L}^2(\Omega) \to E_h \subseteq \mathbf{L}^2(\Omega)$: for all $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $T_h\mathbf{f} = \mathbf{u}_h \in E_h$, where $\mathbf{u}_h$ is the first component of the solution of the problem:

$$\text{find } (\mathbf{u}_h, p_h) \in E_h \times Q_h, \text{ such that}$$
$$\begin{cases} a(\mathbf{u}_h,\mathbf{v}_h) + b(p_h,\mathbf{v}_h) = (\mathbf{f},\mathbf{v}_h) & \forall \mathbf{v}_h \in E_h \\ \overline{b(q_h,\mathbf{u}_h)} = 0 & \forall q_h \in Q_h. \end{cases} \tag{13}$$

We recall that for compact and self-adjoint operator like T, a sufficient and necessary condition in order to have the convergence of the spectrum is the uniform convergence in the operator norm, that is:

$$\lim_{h\to 0} \|T_h - T\|_{\mathcal{L}(\mathbf{L}^2(\Omega),\mathbf{L}^2(\Omega))} = 0. \tag{14}$$

Let us introduce the following assumptions on the finite element space, see [3] for the abstract setting.

H1 *Ellipticity on the discrete kernel* - There exists $\alpha > 0$, independent of h, such that

$$a(\mathbf{u}_h, \mathbf{u}_h) \geq \alpha ||\mathbf{u}_h||^2_{\mathrm{curl}} \quad \forall \mathbf{u}_h \in \mathbb{K}_h \tag{H1}$$

where the discrete kernel $\mathbb{K}_h$ is defined as

$$\mathbb{K}_h = \{\mathbf{u}_h \in E_h \text{ such that } b(q_h, \mathbf{u}_h) = 0 \; \forall q_h \in Q_h\}.$$

H2 *Weak approximability* - There exists $\rho_1(h)$, tending to zero as h goes to zero, such that for any $p \in H^1_{\mathrm{p}}(\Omega)$ it holds

$$\sup_{\mathbf{v}_h \in \mathbb{K}_h} \frac{b(p, \mathbf{v}_h)}{||\mathbf{v}_h||_{\mathrm{curl}}} \leq \rho_1(h) ||p||_1. \tag{H2}$$

H3 *Strong approximability* - There exists $\rho_2(h)$, tending to zero as h goes to zero, such that, for all $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl}; \Omega) \cap (H^{r+1}(\Omega))^3$ with $\mathbf{u} \in \mathbb{K}$, there exists $\mathbf{u}^I \in \mathbb{K}_h$ satisfying

$$||\mathbf{u} - \mathbf{u}^I||_{\mathrm{curl}} \leq \rho_2(h) ||\mathbf{u}||_{r+1}. \tag{H3}$$

Then the following theorem has been proved in [3]:

Theorem 2 *Let us assume that assumptions* H1-H3 *are verified. Then the sequence T_h converges uniformly to T in $\mathcal{L}(\mathbf{L}^2(\Omega), \mathbf{H}_{\mathrm{p}}(\mathrm{curl}; \Omega))$, that is there exists $\rho_3(h)$, tending to zero as h goes to zero, such that*

$$||T\mathbf{f} - T_h\mathbf{f}||_{\mathrm{curl}} \leq \rho_3(h) ||\mathbf{f}||_0, \quad \textit{for all } \mathbf{f} \in \mathbf{L}^2(\Omega). \tag{15}$$

4 Finite element spaces and convergence

Following the ideas of [6], we introduce a modification of standard edge elements satisfying the commuting diagram property with respect to the differential operators $\boldsymbol{\nabla}_\alpha$, $\boldsymbol{\nabla}_\alpha \times$ and $\boldsymbol{\nabla}_\alpha \cdot$. The commuting diagram property and suitable approximation properties will be needed in order to verify that assumptions H1-H3 are fulfilled.

Let $\mathcal{T}_h$ be a triangulation of Ω. For simplicity we consider a mesh of tetrahedra and lowest order Nédélec elements of the first type [9]. The general case will be analyzed in a forthcoming paper.

Let us define the following finite element spaces:

$$
\begin{aligned}
Q_h &= \{q \in H^1_{\mathrm{p}}(\Omega) : \ q|_K = e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{q}, \ \text{for some } \tilde{q} \in \mathcal{P}_1(K) \ \forall K \in \mathcal{T}_h\} \\
E_h &= \{\mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) : \ \mathbf{v}|_K = e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{\mathbf{v}}, \ \text{for some } \tilde{\mathbf{v}} \in \mathcal{E}_0(K) \ \forall K \in \mathcal{T}_h\} \\
F_h &= \{\mathbf{v} \in \mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega) : \ \mathbf{v}|_K = e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{\mathbf{v}}, \ \text{for some } \tilde{\mathbf{v}} \in \mathcal{F}_0(K) \ \forall K \in \mathcal{T}_h\} \\
S_h &= \{v \in \mathbf{L}^2(\Omega) : \ v|_K = e^{-i\boldsymbol{\alpha}\cdot\mathbf{x}}\tilde{v}, \ \text{for some } \tilde{v} \in \mathcal{P}_0(K) \ \forall K \in \mathcal{T}_h\}
\end{aligned}
\tag{16}
$$

where $\mathcal{P}_k(K)$ is the set of the restrictions to K of polynomials of degree less than or equal to k; the elements of $\mathcal{E}_0(K)$ have the form $\mathbf{a}+\mathbf{b}\times\mathbf{x}$ with $\mathbf{a},\mathbf{b} \in \mathbb{C}^3$; the space $\mathcal{F}_0(K)$ contains the vector fields of the form $\mathbf{a}+b\mathbf{x}$, with $\mathbf{a} \in \mathbb{C}^3$ and $b \in \mathbb{C}$.

Let us define the interpolation operators onto the finite element spaces defined above. The degrees of freedom for the space Q_h are again the nodal values, hence the interpolation operator Π^Q_h is the usual nodal interpolation operator.

The edge interpolation operator Π^E_h associates to each function $\mathbf{v}$ of $H^s(\Omega)^3$ the element $\Pi^E_h\mathbf{v} \in E_h$ using the following degrees of freedom on the tetrahedron $K \in \mathcal{T}_h$:

$$
\int_e e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_e)}(\mathbf{v}-\Pi^E_h\mathbf{v})\cdot\mathbf{t}\,ds = 0 \qquad \forall e \text{ edge of } K, \tag{17}
$$

where $\mathbf{x}_e$ is the barycenter of e and $\mathbf{t}$ its tangential unit vector.

Analogously, the face interpolation operator Π^F_h associates to any smooth enough vectorfield $\mathbf{v}$ a discrete element $\Pi^F_h\mathbf{v} \in F_h$ by using the following degrees of freedom on the tetrahedron $K \in \mathcal{T}_h$:

$$
\int_f e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_f)}(\mathbf{v}-\Pi^F_h\mathbf{v})\cdot\mathbf{n}\,ds = 0 \qquad \forall f \text{ face of } K, \tag{18}
$$

where $\mathbf{x}_f$ is the barycenter of f and $\mathbf{n}$ its normal unit vector.

At the end, the degrees of freedom used in order to define the interpolation operator Π^S_h are:

$$
\int_K e^{i\boldsymbol{\alpha}\cdot(\mathbf{x}-\mathbf{x}_K)}(v-\Pi^S_h v)\,ds = 0 \qquad \forall K \in \mathcal{T}_h, \tag{19}
$$

where $\mathbf{x}_K$ is the barycenter of K.

Adapting the proofs of the analogous results in [6], we can prove that the finite element spaces defined in (16) enjoy the following properties:

Lemma 4 *The spaces Q_h, E_h, F_h and S_h satisfy the commuting diagram property:*

$$
\begin{array}{ccccccccc}
0 \rightarrow & Q & \xrightarrow{\boldsymbol{\nabla}_\alpha} & E & \xrightarrow{\boldsymbol{\nabla}_\alpha \times} & F & \xrightarrow{\boldsymbol{\nabla}_\alpha \cdot} & S/\mathbb{R} & \rightarrow 0 \\
& \downarrow \Pi_h^Q & & \downarrow \Pi_h^E & & \downarrow \Pi_h^F & & \downarrow \Pi_h^S & \\
0 \rightarrow & Q_h & \xrightarrow{\boldsymbol{\nabla}_\alpha} & E_h & \xrightarrow{\boldsymbol{\nabla}_\alpha \times} & F_h & \xrightarrow{\boldsymbol{\nabla}_\alpha \cdot} & S_h/\mathbb{R} & \rightarrow 0
\end{array}
\tag{20}
$$

In the above diagram, the spaces Q, E, F, S are suitable smooth dense subspaces of $H^1_{\mathrm{p}}(\Omega)$, $\mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega)$, $\mathbf{H}_{\mathrm{p}}(\mathrm{div};\Omega)$ and $L^2(\Omega)$, respectively.

Lemma 5 *There exist C, independent of h, such that the following interpolation error estimates hold true for sufficiently regular functions:*

$$\|q - \Pi_h^Q q\|_1 \leq C h^{r-1} \|q\|_r \quad 1 \leq r \leq 2, \tag{21}$$
$$\|\mathbf{v} - \Pi_h^E \mathbf{v}\|_0 \leq C h^s \left(|\mathbf{v}|_s + \|\boldsymbol{\nabla} \times \mathbf{v}\|_s\right) \quad 1/2 < s \leq 1, \tag{22}$$
$$\|\boldsymbol{\nabla} \times \mathbf{v} - \boldsymbol{\nabla} \times \Pi_h^E \mathbf{v}\|_0 \leq C h^t |\boldsymbol{\nabla} \times \mathbf{v}|_t \quad 0 < t \leq 1, \tag{23}$$
$$\|\mathbf{v} - \Pi_h^F \mathbf{v}\|_0 \leq C h^s \left(|\mathbf{v}|_s + \|\boldsymbol{\nabla} \cdot \mathbf{v}\|_s\right) \quad 1/2 < s \leq 1, \tag{24}$$
$$\|\boldsymbol{\nabla} \cdot \mathbf{v} - \boldsymbol{\nabla} \cdot \Pi_h^F \mathbf{v}\|_0 \leq C h^t |\boldsymbol{\nabla} \cdot \mathbf{v}|_t \quad 0 < t \leq 1, \tag{25}$$
$$\|v - \Pi_h^S v\|_0 \leq C h^r \|v\|_r \quad 0 < r < 1. \tag{26}$$

A consequence of Lemma 4 is the following discrete version of theorem 1.

Lemma 6 *Let $\mathbf{u}_h \in E_h$, then there exist $\mathbf{z}_h \in E_h$ and $q_h \in Q_h$ such that*

$$\mathbf{u}_h = \mathbf{z}_h + \boldsymbol{\nabla}_\alpha q_h, \qquad \text{and } b(q_h, \mathbf{z}_h) = 0, \tag{27}$$

where $\mathbf{z}_h \in E_h$ can be characterized by means of the following mixed problem:

$$
\begin{array}{l}
\text{find } (\mathbf{z}_h, \boldsymbol{\sigma}_h) \in E_h \times F_h \text{ such that} \\
\left\{
\begin{array}{ll}
(\mathbf{z}_h, \mathbf{w}_h) + (\boldsymbol{\sigma}_h, \boldsymbol{\nabla}_\alpha \times \mathbf{w}_h) = 0 & \forall \mathbf{w}_h \in E_h \\
(\boldsymbol{\tau}_h, \boldsymbol{\nabla}_\alpha \times \mathbf{z}_h) = (\boldsymbol{\tau}_h, \boldsymbol{\nabla}_\alpha \times \mathbf{u}_h) & \forall \boldsymbol{\tau}_h \in F_h.
\end{array}
\right.
\end{array}
\tag{28}
$$

We have now all the elements which will be needed for the proof of the assumptions H1-H3. This will be done in the next three lemmas, whose proofs is briefly sketched. The complete details will be reported in a forthcoming paper.

Lemma 7 *There exists a constant C not depending on h such that for all $\mathbf{u}_h \in \mathbb{K}_h$ it holds*

$$\|\mathbf{u}_h\|_0 \leq C \|\boldsymbol{\nabla}_\alpha \times \mathbf{u}_h\|_0. \tag{29}$$

Proof. The proof follows the same lines as the analogous one for the $\nabla\times$ operator given in [1]. Due to Lemma 6 there exist $\mathbf{z}_h \in E_h$ and $q_h \in Q_h$ such that

$$\mathbf{u}_h = \mathbf{z}_h + \nabla_\alpha q_h, \qquad \text{with } b(q_h, \mathbf{z}_h) = 0.$$

Moreover, since $E_h \subseteq \mathbf{H}_\mathrm{p}(\mathrm{curl};\Omega)$, there exists $\mathbf{z} \in (H^1_\mathrm{p}(\Omega))^3$ and $q \in H^1_\mathrm{p}(\Omega)$ such that the following orthogonal decomposition of $\mathbf{u}_h$ holds true (see Theorem 1)

$$\mathbf{u}_h = \mathbf{z} + \nabla_\alpha q \quad \text{with } b(q, \mathbf{z}) = 0.$$

Here $\mathbf{z}$ satisfies a mixed problem analogous to (28), with datum $\nabla_\alpha\times \mathbf{u}_h$, with the following a priori estimate

$$||\mathbf{z}||_0 \leq C||\nabla_\alpha\times \mathbf{u}_h||_0.$$

Thanks to the commuting diagram property (20) and the regularity properties, see Lemma 3, one obtains

$$||\mathbf{z} - \mathbf{z}_h||_0 \leq Ch^s||\nabla\times \mathbf{u}_h||_0.$$

Due to the orthogonality of $\mathbf{u}_h$ with $\nabla_\alpha q_h$ we have:

$$||\mathbf{u}_h||_0^2 = (\mathbf{u}_h, \mathbf{u}_h) = (\mathbf{u}_h, \mathbf{z}_h) = (\mathbf{u}_h, \mathbf{z}_h - \mathbf{z}) + (\mathbf{u}_h, \mathbf{z}) \leq C||\mathbf{u}_h||_0(||\mathbf{z} - \mathbf{z}_h||_0 + ||\mathbf{z}||_0).$$

Using the last two inequalities one gets the desired bound (29). $\square$
Let us now verify that assumption H2 holds true.

Lemma 8 *For all $\mathbf{v}_h \in \mathbb{K}_h$ there exists $\mathbf{v} \in \mathbb{K}$ such that*

$$||\mathbf{v}_h - \mathbf{v}||_0 \leq Ch^s||\mathbf{v}_h||_\mathrm{curl}, \quad \textit{with } s > 1/2.$$

Proof. The proof uses essentially the same tools as the previous one. In fact, $\mathbf{v}_h$ is the solution of a problem like (28) with datum $\nabla_\alpha\times \mathbf{v}_h$. It is enough to take $\mathbf{v}$ as the solution of the corresponding continuous problem. $\square$

In order to prove H2, we write:

$$\sup_{\mathbf{v}_h\in\mathbb{K}_h} \frac{b(p, \mathbf{v}_h)}{||\mathbf{v}_h||_\mathrm{curl}} = \sup_{\mathbf{v}_h\in\mathbb{K}_h} \frac{b(p, \mathbf{v}_h - \mathbf{v})}{||\mathbf{v}_h||_\mathrm{curl}} \leq Ch^s||p||_1,$$

where $\mathbf{v}$ is given by Lemma 8.

It remains to prove H3.

Lemma 9 *There exists a constant C, such that for all $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap (H^{r+1}(\Omega))^3$ with $\mathbf{u} \in \mathbb{K}$, there is an element $u^I \in \mathbb{K}_h$ satisfying*

$$\|\mathbf{u} - \mathbf{u}^I\|_{\mathrm{curl}} \leq Ch^r \|\mathbf{u}\|_{r+1}. \tag{30}$$

Proof. Let us consider $\mathbf{u} \in \mathbf{H}_{\mathrm{p}}(\mathrm{curl};\Omega) \cap (H^{r+1}(\Omega))^3$ with $\mathbf{u} \in \mathbb{K}$. Then there exists $\mathbf{g} \in \mathbf{H}_{\mathrm{p}}(\mathrm{div}^0_\alpha;\Omega)$ such that the couple $(\mathbf{u}, p = 0)$ is solution of problem (8) with datum $\mathbf{g}$. Let us take as $\mathbf{u}^I$ the first component of the solution of (13) with the same datum. We can adapt to this problem the known error estimates for the standard Maxwell equations, see e.g. [8] and we obtain

$$\|\mathbf{u} - \mathbf{u}^I\|_{\mathrm{curl}} \leq C \inf_{\mathbf{v}_h \in E_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathrm{curl}}, \tag{31}$$

then the interpolation error estimates (22) and (23) give (30). □

As a consequence of the results of [3, 2], we have proved the following theorem:

Theorem 3 *There exists a constant C such that for all $\mathbf{f} \in \mathbf{L}^2(\Omega)$ it holds*

$$\|T\mathbf{f} - T_h\mathbf{f}\|_{\mathrm{curl}} \leq Ch^t \|\mathbf{f}\|_0 \tag{32}$$

where $t = \inf(s, r)$, and s and r are given in Lemma 3.

Let ω_i^2 be an eigenvalue of problem (5), with multiplicity m_i and denote by E_i the corresponding eigenspace. Then, due to (32), exactly m_i discrete eigenvalues $\omega^2_{i_1,h}, \dots, \omega^2_{i_{m_i},h}$ converge to ω_i^2. Moreover, setting $\hat{\omega}^2 i, h = (1/m_i)\sum_{j=1}^{m_i} \omega^2 i_j, h$ and denoting by $\hat{E}_{h,i}$ the direct sum of the eigenspaces corresponding to $\omega^2 i_1, h, \dots, \omega^2 i_{m_i}, h$, we have that there exists h_0 such that for $0 < h < h_0$ the following inequalities hold:

$$\begin{aligned} |\omega_i^2 - \hat{\omega}^2 i, h| &\leq Ce_h^{2t} \\ \delta(E_i, \hat{E}_{h,i}) &\leq Ce_h^t, \end{aligned} \tag{33}$$

where $\delta(E_i, \hat{E}_{h,i})$ denotes the gap between E_i and $E_{h,i}$.

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