# Finite element method in 1D

## Non homogeneous Dirichlet boundary conditions
## Neumann boundary conditions
## Advection diffusion equation

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# Outline

## Non homogeneous Dirichlet boundary conditions

**Problem** Find $u$ such that

$$-\mu u'' + \sigma u = f \quad \text{in } ]a, b[ \qquad u(a) = \alpha, \ u(b) = \beta$$

Let us consider $Rg \in H^1$ such that $Rg(a) = \alpha$ and $Rg(b) = \beta$

(e.g. $Rg(x) = \alpha \dfrac{b-x}{b-a} + \beta \dfrac{x-a}{b-a}$) and $u_0 = u - Rg$. Hence

$u_0 \in H_0^1(a, b)$ solves:

$$- \mu u_0'' + cu_0 = -\mu u'' + cu + \mu Rg'' - cRg = f + \mu Rg'' + cRg$$
$$u_0(a) = u_0(b) = 0.$$

### Weak formulation

Find $u_0 \in H_0^1(a, b)$ such that

$$a(u, v) = F(v) - a(Rg, v) \quad \forall v \in H_0^1(a, b),$$

where $\quad a(u, v) = \displaystyle\int_a^b (\mu u' v' + \sigma uv)\, dx, \quad F(v) = \int_a^b fv\, dx$

## Non homogeneous Dirichlet boundary conditions (in practice)

We consider $Rg_h$ such that $Rg_h(a) = \alpha$ and $Rg_h(b) = \beta$.
E.g.: $Rg_h(x) = \alpha \varphi_a(x) + \beta \varphi_b(x)$ with $\varphi_a$ and $\varphi_b$ basis functions associated to the end points $a$ and $b$, respectively.

$$a(Rg_h, \varphi_i) = \alpha a(\varphi_a, \varphi_i) + \beta a(\varphi_b, \varphi_i) = B\vec{Rg}.$$

Here $B \in \mathbb{R}^{ndof \times 2}$ with elements

$$B_{ij} = \mu \int_a^b \varphi_j' \varphi_i' \, dx + \sigma \int_a^b \varphi_j \varphi_i \, dx \quad j = a, b \text{ e } i = 1, \dots, ndof.$$

We compute $u_{0h} \in V_h$ as follows:

$$A\vec{u_0} = b - B\vec{Rg}, \qquad \vec{u} = \vec{u_0} + \vec{Rg}.$$

Notice that the vector $-B\vec{Rg}$ has only the first and the last components different from zero with values

$$-B\vec{Rg}_1 = \alpha(\mu/h + h\sigma), \quad -B\vec{Rg}_{ndof} = \beta(\mu/h + h\sigma).$$

# Exercize 5

Consider the following differential equation

$$-u'' + 2u = 4e^x \sin(x) \quad x \in (0, 2\pi) \quad u(0) = 1, \quad u(2\pi) = e^{2\pi}.$$

▸ The function `femP1` is a simplified version of `fem1d` and computes the solution only with linear piecewise polynomials. Modify the function `femP1` in order to take into account non homogeneous boundary conditions.

▸ Solve the differential equation by means of the modified function `femP1` with `N=[10 20 40 80 160 320]`.

▸ Compute the relative error in the $L^2$-norm of the function and of its derivative using the function `errore`.

▸ Plot the error in a bilogarithmic scale.

# Neumann boundary conditions

Let us consider the following differential equation

$$-\mu u'' + \sigma u = f \quad \text{in } [a, b]$$
$$u'(a) = \alpha, \qquad u'(b) = \beta$$

We multiply the equation by $v \in H^1(a, b)$ and integrate on $(a, b)$

$$\int_a^b (-\mu u'' + \sigma u)v dx = [-\mu u' v]_a^b + \int_a^b (\mu u' v' + \sigma uv) dx$$

$$= -\mu u'(b)v(b) + \mu u'(a)v(a) + \int_a^b (\mu u' v' + \sigma uv) dx$$

$$= -\mu \beta v(b) + \mu \alpha v(a) + \int_a^b (\mu u' v' + \sigma uv) dx = \int_a^b fv dx$$

## Weak formulation

Find $u \in H^1(a, b)$ such that

$$a(u, v) = F(v) + \mu \beta v(b) - \mu \alpha v(a) \quad \forall v \in H^1(a, b).$$

## Discretizzazione con elementi finiti lineari

We subdivide $[a, b]$ into $N$ intervals with size $h = (b - a)/N$

$$V_h = \{v \in H^1(a, b) : v|_{I_i} \in \mathbb{P}^1\}$$

hence we have $N + 1$ basis functions $\varphi_i$ including those associated to the end points. The matrix are:

$$K = \frac{1}{h} \begin{bmatrix} 1 & -1 & \ldots & \ldots & \ldots & \ldots & 0 \\ -1 & 2 & -1 & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & -1 & 2 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & -1 & 2 & -1 \\ 0 & \ldots & \ldots & \ldots & \ldots & -1 & 1 \end{bmatrix}$$

$$M = \frac{h}{6} \begin{bmatrix} 2 & 1 & \ldots & \ldots & \ldots & \ldots & 0 \\ 1 & 4 & 1 & \ldots & \ldots & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 4 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \ldots & \ldots & 1 & 4 & 1 \\ 0 & \ldots & \ldots & \ldots & \ldots & 1 & 2 \end{bmatrix}$$

▶ If $\sigma > 0$ we find the solution by solving the linear system $(\mu K + \sigma M)u = \underline{F}$.

▶ If $\sigma = 0$ the following compatinility condition is necessary for the solvability of the system

$$\int_a^b f dx + \mu\beta - \mu\alpha = 0$$

and we compute a solution with zero mean value.
In practice we assign the value of $u$ in an internal node of the mesh and then we correct the mean value.

# Esercizio 6
$\sigma > 0$

Si considerino le equazioni differenziali:

$$-u''(x) + 2u(x) = e^x \quad \text{per } x \in [0,1]$$
$$u'(0) = 1, \quad u'(1) = e$$
$$u(x) = e^x.$$

$$-u''(x) + u(x) = -2 + 12x - 12x^2 + x^2(x-1)^2 \text{ per } x \in [0,1]$$
$$u'(0) = 0, \quad u'(1) = 0$$
$$u(x) = x^2(x-1)^2.$$

Risolvere le equazioni date con i seguenti valori di
$$\texttt{N=[8 16 32 64 128 256 512]}$$
usando la function `femP1N`.
Calcolare l'errore usando la function `errore`.nframe

# Esercizio 7
$\sigma = 0$

Si considerino le equazioni differenziali:

$$-u''(x) = -e^x \quad \text{per } x \in [0,1]$$
$$u'(0) = 1, \quad u'(1) = e$$
$$u(x) = e^x - e + 1.$$

$$-u''(x) = -2 + 12x - 12x^2 \text{ per } x \in [0,1]$$
$$u'(0) = 0, \quad u'(1) = 0$$
$$u(x) = x^2(x-1)^2 - 1/30.$$

Risolvere le equazioni date con i seguenti valori di
N=[8 16 32 64 128 256 512]
usando la function femP1N.
Calcolare l'errore usando la function errore.

## Equazione di diffusione, trasporto e reazione

Equazione differenziale completa

$$-\mu u'' + \beta u' + cu = f \quad \text{in } (a, b)$$
$$u(a) = u_a, \quad u(b) = u_b$$

### Problema debole

Trovare $u \in H_0^1(a, b)$ tale che

$$\int_a^b (\mu u'v' + \beta u'v + cuv) \, dx = \int_a^b fv \, dx \quad \forall v \in H_0^1(a, b)$$

## Costruzione delle matrici

Resta da calcolare la matrice associata al termine del primo ordine. Suddiviso l'intervallo $[a, b]$ in $N$ parti uguali di ampiezza $h = (b - a)/N$, la matrice non simmetrica associata al termine del primo ordine è

$$B = \frac{1}{2} \begin{bmatrix} 0 & 1 & \dots & \dots & \dots & \dots & 0 \\ -1 & 0 & 1 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & -1 & 0 & 1 \\ 0 & \dots & \dots & \dots & \dots & -1 & 0 \end{bmatrix}$$

## Esercizio 8

Si consideri l'equazione differenziale

$$-u'' + 2u' = f \quad x \in (0,1) \qquad u(0) = u_a, \ u(1) = u_b$$

con i seguenti dati:

$$f_1(x) = 4 - 4x, \ u_a = u_b = 0$$
$$f_2(x) = -2(1-x)^2 - 2x^2 + 4 + 4x(1-x)(3-2x), \ u_a = 0, \ u_b = 2$$
$$f_3(x) = (4\pi^2 - 4\pi)\sin(2\pi x) + (4\pi^2 + 4\pi)\cos(2\pi x) \ u_a = u_b = 1$$
$$f_4(x) = e^x(1-x) \ u_a = 1, \ u_b = 0$$

Risolvere l'equazione differenziale con elementi finiti lineari e quadratici usando la function `fem1d` al variare di N=[10 20 40 80 160 320].
Calcolare l'errore relativo in norma $L^2$ sia della funzione che della derivata usando la function `errore`.
Riportare opportunamente i risultati in scala bilogaritmica.

## Soluzioni dell'esercizio 8

La soluzione esatta dell'equazione differenziale dell'esercizio 1 ha la seguente espressione analitica:

$$u_1(x) = x(1-x)$$
$$u_2(x) = x^2(1-x)^2 + 2x$$
$$u_3(x) = \sin(2\pi x) + \cos(2\pi x)$$
$$u_4(x) = e^x(1-x)$$

## Problemi con trasporto dominante

Si consideri l'equazione differenziale:

$$-\varepsilon u''(x) + u'(x) = 0 \quad \text{per } x \in [0, 1]$$
$$u(0) = 0, \quad u(1) = 1.$$

Risolvere l'equazione data con i seguenti valori di N=[10 20 40 80 160], mediante il metodo degli elementi finiti, per i seguenti valori $\varepsilon = 1$, $\varepsilon = 0.1$ e $\varepsilon = 0.01$.