Hyperbolic Partial Differential Equations

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Outline

Linear transport equation on $\mathbb{R}$

Problem

Find $c(x,t) : \mathbb{R} \times [0,T] \to \mathbb{R}$ such that

$$\frac{\partial c}{\partial t} + a\frac{\partial c}{\partial x} = 0 \quad x \in \mathbb{R},\ t \in (0,T]$$
$$c(x,0) = c_0(x) \quad x \in \mathbb{R}.$$

Characteristic lines

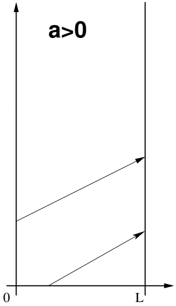
For all $x_0 \in \mathbb{R}$, we consider the ordinary differential equation

$$\frac{dx}{dt}(t) = a,\ t \in (0,T], \qquad x(0) = x_0.$$

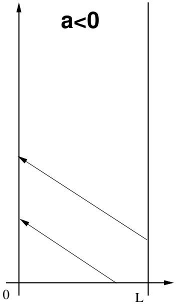
The curves $x(t)$ are the characteristic lines of the transport equation.

Exact solution $c(x,t) = c_0(x - at)$

Inflow boundary



For $a > 0$ the characteristic lines propagate from the left to the right.
inflow boundary $x_{in} = 0$.



For $a < 0$ the characteristic lines propagate from the right to the left.
inflow boundary $x_{in} = L$.

Linear transport equation on bounded domains

Problem

Find $c(x,t) : [0,L] \times [0,T] \to \mathbb{R}$ such that

$$
\begin{aligned}
&\frac{\partial c}{\partial t} + a\frac{\partial c}{\partial x} = 0 \quad x \in (0,L),\ t \in (0,T] \\
&c(x,0) = c_0(x) \quad x \in (0,1) \\
&c(x_{in},t) = c_1(t) \quad t \in (0,T].
\end{aligned}
$$

Exact solution

$$
c(x,t) = \begin{cases} c_0(x - at) & \text{if } 0 < x - at < L \\ c_1(t - x/a) & \text{if } x - at < 0 \text{ or } x - at > L \end{cases}
$$

The finite difference method

- time step Δt
- mesh size h (for bounded domains $h = L/N$)
- grid points $(x_j, t^n) = (jh, n\Delta t)$
- discrete solution $c_j^n \approx c(x_j, t^n)$

We set:

$$\lambda = \Delta t / h$$
$$x_{j+1/2} = x_j + h/2$$

Finite difference method

$$c_j^{n+1} = c_j^n - \lambda(H_{j+1/2}^n - H_{j-1/2}^n)$$

with $H_{j+1/2}^n = H(c_j^n, c_{j+1}^n)$.
The function $H(\cdot,\cdot)$ is the *numerical flux*.

CFL condition

$$|\lambda a| \leq 1$$

Forward Euler/Centered – FE/C

$$c_j^{n+1} = c_j^n - \frac{\lambda}{2} a(c_{j+1}^n - c_{j-1}^n)$$
$$H_{j+1/2} = \frac{1}{2} a(c_{j+1} + c_j)$$

Truncation error

$$\tau(\Delta t, h) = \mathcal{O}(\Delta t + h^2).$$

Stability

FE/C is stable, that is

$$\|c^n\|_{\Delta,2} \leq e^{T/2} \|c^0\|_{\Delta,2}$$

under the condition

$$\Delta t \leq \left(\frac{h}{a}\right)^2.$$

Lax-Friedrichs – LF

$$c_j^{n+1} = \frac{1}{2}(c_{j+1}^n + c_{j-1}^n) - \frac{\lambda}{2} a(c_{j+1}^n - c_{j-1}^n)$$
$$H_{j+1/2} = \frac{1}{2}\big(a(c_{j+1} + c_j) - \lambda^{-1}(c_{j+1} - c_j)\big)$$

Truncation error

$$\tau(\Delta t, h) = \mathcal{O}\Big(\Delta t + h^2 + \frac{h^2}{\Delta t}\Big).$$

Stability

If the CFL condition is satisfied, LF is strongly stable

$$\|c^n\|_{\Delta,1} \leq \|c^{n-1}\|_{\Delta,1}.$$

Lax-Wendroff – LW

$$c_j^{n+1} = c_j^n - \frac{\lambda}{2} a(c_{j+1}^n - c_{j-1}^n) + \frac{\lambda^2 a^2}{2}(c_{j+1}^n - 2c_j^n + c_{j-1}^n)$$
$$H_{j+1/2} = \frac{1}{2}\left(a(c_{j+1} + c_j) - \lambda a^2(c_{j+1} - c_j)\right)$$

Truncation error

$$\tau(\Delta t, h) = \mathcal{O}\Big(\Delta t^2 + h^2 + h^2 \Delta t\Big).$$

Stability

Under the CFL condition, LW is strongly stable:
$\|c^n\|_{\Delta,2} \leq \|c^{n-1}\|_{\Delta,2}$.
Using the von Neumann analysis: if $c_j^0 = c_0(x_j) = \sum_{k=-\infty}^{\infty} \alpha_k e^{ikjh}$, then

$$c_j^n = \sum_{k=-\infty}^{\infty} \alpha_k e^{ikjh} \gamma_k^n \text{ with } |\gamma_k| = 1 - 4\lambda^2 a^2 \sin^4\left(\frac{hk}{2}\right)(1 - \lambda^2 a^2).$$

Upwind – U

$$c_j^{n+1} = c_j^n - \frac{\lambda}{2} a(c_{j+1}^n - c_{j-1}^n) + \frac{\lambda}{2}|a|(c_{j+1}^n - 2c_j^n + c_{j-1}^n)$$

$$H_{j+1/2} = \frac{1}{2}\big(a(c_{j+1} + c_j) - |a|(c_{j+1} - c_j)\big)$$

Truncation error

$$\tau(\Delta t, h) = \mathcal{O}(\Delta t + h).$$

Stability

If the CFL condition is satisfied, U is strongly stable

$$\|c^n\|_{\Delta,1} \leq \|c^{n-1}\|_{\Delta,1}.$$

Function for solving linear transport equation

Input

- `a` propagation rate;
- `I` space interval, `T` final time;
- `u0, u1` initial and inflow data;
- `N` number of subdivision of the interval $[0, L]$;
- `lambda` $\lambda = \Delta t/h$.

Output

- `x` grid points;
- `t` time;
- `u` n-th row contains the values of c in (x, t^n).

Exercise 1

Consider the equation:

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = 0 \quad x \in (-2,3), t \in (0,1.6]$$

$$c(x,0) = \begin{cases} 1-|x| & |x| \leq 1 \\ 0 & |x| \geq 1 \end{cases}$$

$$c(-2,t) = 0 \quad t \in (0,1.6]$$

- Solve the equation using LF with $h = 0.1$ and $\lambda = 0.8$.
- Compare the computed solution with the exact one.
- Use smaller values for h and the same value for λ.
- Compute the solution for $T = 0.8$ with the same values of h and $\lambda = 1.6$.
- Compute the solution with the other schemes and compare the computed solutions.

Exercise 2

For values of x in the interval $[-1, 3]$ and t in $[0, 2.4]$, solve the transport equation

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = 0,$$

with the initial data

$$c(x,0) = \begin{cases} \cos^2(\pi x) & |x| \leq 1/2 \\ 0 & otherwise \end{cases}$$

and the boundary data $c(-1, t) = 0$.
Use the four schemes for $h = 1/10$, $h = 1/20$, and $h = 1/40$ as follows

a. Upwind with $\lambda = 0.8$

b. FE/C with $\lambda = 0.8$

c. LF with $\lambda = 0.8$ and $\lambda = 1.6$

d. LW with $\lambda = 0.8$

How does the error decrease as the mesh gets finer?

Exercise 3

For values of x in the interval $[0, 10]$ and t in $[0, 10]$, solve the transport equation

$$\frac{\partial c}{\partial t} + \frac{\partial c}{\partial x} = 0,$$

with the initial data

$$c(x, 0) = \begin{cases} \sin(2\pi x) & 0 \le x \le 1 \\ 0 & \textit{otherwise} \end{cases}$$

and the boundary data $c(0, t) = 0$.